$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point Q (in the positive direction of rotation) at the instant in question, and $d\omega/dt$ = the angular acceleration of the body at that instant (θ being measured in radians, in the positive direction of rotation, and $\omega = d\theta/dt$); while k = the radius of gyration of the body about the point Q (that is, about an axis through Q perpendicular to the plane of the motion).

Here by a "suitably chosen" point Q we mean any point Q of the body such that (1) Q is fixed in the plane; or (2) Q is moving with uniform velocity in a straight line; or (3) Q is the center of mass; or (4) Q has an acceleration whose direction, at the instant in question, passes through the center of mass; and the equation above will be valid when and only when Q has some one of these four properties.

Proof of Case 1. Entirely elementary proofs for the case of rotation about a fixed axis are readily accessible and need not be repeated here. We shall mention only that this case can be conveniently recalled to mind when stated in the following form: If a rigid body is free to rotate about a fixed axis, under the action of any turning moment, the angular acceleration at any instant is the same as if all the material of the body were concentrated in a single particle at a distance k from the axis, where k is the radius of gyration of the body about the axis.

Proof of Case 2. First, regard the given body as a system of free particles acted on by definite external and internal forces, under given initial conditions (the internal forces being just sufficient to keep the system rigid); and let V_x, V_y be the initial values of the velocity components of the given point Q, referred to fixed axes in the plane.

Now imagine the initial conditions so changed that the initial velocity components of each and every particle are diminished by the amounts V_x and V_y, all the forces remaining the same as before. In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular acceleration will remain unchanged; but the point Q will be reduced to rest.

Under these new conditions, therefore, we may apply the theorem of case 1, thus obtaining the equation $\Sigma Fp = (W/g)k^2 d\omega/dt$, where all the letters have their original meanings. The truth of the equation is thus proved for case 2.

Proof of Case 4, which includes case 3. Regard the given body again as a system of free particles, as in case 2, and let a_x and a_y be the acceleration components of the given point Q, referred to fixed axes in the plane.

Now imagine impressed upon each and every particle w an additional force with components $-(w/g)a_x$ and $-(w/g)a_y$; this will clearly have the effect of diminishing the acceleration components of each particle by the amounts a_x and a_y. In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular

acceleration will remain unchanged; but the point Q will be reduced to a state of uniform motion in a straight line.

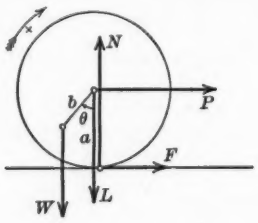
Under these new conditions, therefore, we may apply the theorem of case 2, and thus obtain the equation

$$\Sigma Fp + \Sigma f'p' = (W/g)k^2 d\omega/dt,$$

where ΣFp and ω have their original meanings, and $\Sigma f'p'$ = the sum of the moments about Q of the additional impressed forces.

Now it will be noticed that these additional forces form, at each instant, a set of parallel forces each of which is proportional to the mass of the particle on which it acts. Hence, by the definition of the center of mass, in finding the sum of the moments of these parallel forces, we may replace the entire set of forces by a single force F' acting at the center of mass. Furthermore, the direction of these parallel forces, and hence the direction of F', is opposite to the direction of the vector acceleration of the point Q; but, by the hypothesis of case 4, this direction lies along a line drawn through Q and the center of mass. Hence the moment of F' about Q will be zero; that is, $\Sigma f'p' = 0$, and the theorem is established.

It should be carefully noted that in the statement of theorem I the instantaneous center—that is, the point of no velocity—is not included in the list of "suitable" points Q; in other words, it is not legitimate to "take moments about the instantaneous center," unless this point happens to have also one of the other properties explicitly mentioned in the theorem.



To illustrate the danger of "taking moments about the instantaneous center," consider the following simple problem.

Suppose a carriage is moving with uniform velocity along a level road, and consider the motion of a wheel, of radius a, in which the center of mass does not coincide with the center of the wheel. The center of mass will then move in a trochoidal path, and the components of its acceleration will be $\ddot{x} = b\omega^2 \sin\theta$, $\ddot{y} = b\omega^2 \cos\theta$, where b is the distance from the hub to the center of mass, and ω the constant angular velocity of the wheel.

The forces acting on the wheel are (1) the weight W, acting downward at the center of mass; (2) the pressure of the bearings, which we may suppose resolved

into a forward component P, and a downward component L; and (3) the reaction of the road, which we may suppose resolved into a forward component F, and an upward component N. The equations of motion for the center of mass are therefore

$$P + F = (W/g)b\omega^2 \sin\theta, \tag{1}$$

$$N - W - L = (W/g)b\omega^2 \cos\theta. \tag{2}$$

Since the angular velocity is constant, $d\omega/dt = 0$, and the equation of rotation may be written as follows:
(a) about the center of mass:

$$Pb\cos\theta + (L - N)b\sin\theta - F(a - b\cos\theta) = 0; \tag{3a}$$

(b) about the hub, which is moving with uniform velocity in a straight line:

$$Fa + Wb\sin\theta = 0; \tag{3b}$$

(c) about the lowest point of the wheel, which is the instantaneous center:

$$Pa - Wb\sin\theta = 0. \tag{3c}$$

Of the last three equations, (3a) and (3b) are true, in accordance with theorem I, and from either of them, combined with (1) and (2), we find

$$Pa - Wb\sin\theta = \frac{W}{g}ab\,\omega^2\sin\theta. \tag{4}$$

But this shows that equation (3c) is false. Hence in this case, wholly erroneous results would be obtained by taking moments about the instantaneous center.

If the center of mass had coincided with the hub, it would then indeed have been legitimate to take moments about the lowest point of the wheel, not, however, because this point is the instantaneous center, but because it is in this case a point whose vector acceleration at the instant in question passes through the center of mass (case 4). The use of case 4 and case 2 will often lead to simpler equations that could be obtained if we confined ourselves to the use of the more familiar cases 1 and 3.

Corollary I. In discussing the motion of a rigid body in a plane, any given set of external forces may be replaced by (1) a single force, acting at the center of mass, and (2) a couple, acting anywhere in the body. The components of the single force will be $(W/g)a_x$ and $(W/g)a_y$, where a_x and a_y are the components of the acceleration of the center of mass. The moment of the couple will be $(W/g)k^2d\omega/dt$, where $d\omega/dt$ is the angular acceleration of the body, and k its radius of gyration about the center of mass.

The single force, acting at the center of mass, will have no effect on the angular velocity of the body; while the couple will have no effect on the motion of the center of mass.

It is clear, however, from theorem I, that a single force whose line of action does not pass through the center of mass will produce not only a change in the motion of the center of mass, but also a change in the angular velocity of the body.

Corollary II. In discussing the motion of a rigid body in a plane, the point of application of any force may be shifted at pleasure along the line of action of that force.

For the moment of the force about any point, and the components of the force along any axes will be unchanged.

This corollary, known as the principle of transmissibility of forces, is usually assumed as a fundamental principle of statics. But this assumption is not required for the proof of the theorems here stated.

Theorem II. *Theorem of moments about a point fixed in the plane.* If a rigid body W is free to move in a plane, under the action of any external forces, and if O is any point fixed in the plane, then, at any instant,

$$\Sigma Fp = \frac{W}{g}\frac{d(h\bar{v})}{dt} + \frac{W}{g}\bar{k}^2\frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point O (in the positive direction of rotation) at the instant in question, and

$h\bar{v}$ = the moment about O of the velocity $\bar{v}$ of the center of mass at that instant; while

$\bar{k}$ = the radius of gyration of the body *about its center of mass.*

Proof. Let C be the point of the plane through which the center of mass is passing at the instant in question, and let a_r and a_θ be the acceleration components of the center of mass along and perpendicular to the fixed axis OC at that instant.

By corollary I, the external forces acting on the body may be replaced by (1) a single force, acting at the center of mass, with components $(W/g)a_r$ and $(W/g)a_\theta$ along and perpendicular to OC; and (2) a couple, acting anywhere in the body, with a moment (about any point) equal to $(W/g)\bar{k}^2 d\omega/dt$. Hence the sum of the moments of the external forces about O reduces to

$$\Sigma Fp = (W/g)ra_\theta + (W/g)\bar{k}^2 d\omega/dt,$$

where $r = \overline{OC}$. But from the elementary kinematics of a particle in polar coordinates, $a_\theta = \frac{1}{r}\frac{d(r\bar{v}_\theta)}{dt}$, where $\bar{v}_\theta = \frac{rd\theta}{dt}$ = the velocity component of the center of mass in the direction perpendicular to OC. Also, $r\bar{v}_\theta = h\bar{v}$, where h is the perpendicular from O to the vector $\bar{v}$. Hence

$$\Sigma Fp = (W/g)d(h\bar{v})/dt + (W/g)\bar{k}^2 d\omega/dt,$$

which establishes the theorem.

This theorem II is by no means so simple or so important as theorem I. It is useful chiefly in problems in impact, especially in the derived form obtained by multiplying through by dt and integrating from $t = t_0$ to $t = t_1$. In this derived form the equation is known as the equation of impulse and momentum. But neither of these concepts is required for the proof as here given.

THE NAPIER TERCENTENARY CELEBRATION.

By FLORIAN CAJORI.

The Napier Tercentenary Celebration which was held in Edinburgh, July 24–27, was a marked success, notwithstanding the rather limited attendance of British mathematicians, some of whom had left for the meeting of the British Association for the Advancement of Science in Australia. There were present in Edinburgh a considerable number of Germans, French, Russians, Americans, and a few from other countries. Altogether there were over forty foreigners. It may be worthy of note that when, on July 28, war alarms began to come in, all delegates from the European continent took a hasty departure for their respective countries.

The opening exercises on Friday, July 24, were of a general nature, Lord Moulton delivering the principal address. On Saturday morning the papers relating to the history of logarithms were read; on Monday came research papers on logarithms or allied topics. Saturday afternoon the members were invited to a garden party at Merchiston Castle, for some years the abode of John Napier. It was interesting to see the representative mathematicians going there, making, so to speak, a pilgrimage to a shrine. Lord Moulton's address was delivered in the hall of the Edinburgh Students' Union.

The address of Lord Moulton, one of the Lords of Appeal, was interesting, first of all, because it was a mathematical address written by a great judge. This was not strange at all to those who remembered that in his undergraduate days at Cambridge Lord Moulton distinguished himself in mathematics and later served for a few years as fellow and lecturer. It was he who brought out the second edition of Boole's Calculus of Finite Differences. Lord Moulton's address was a remarkable one. One of his statements near the beginning—that the invention of logarithms came on the world as a bolt from the blue—might indeed be questioned by the historian, but the main part of his address, which endeavored to show the evolution of logarithms in Napier's mind, to reconstitute the process of discovery by deciphering the half-effaced records of its growth, is worthy of most careful study.

The meeting on Saturday morning opened with Professor Hobson of Cambridge in the chair. Dr. J. W. L. Glaisher of Cambridge gave an address on the work of Napier, stating that it was the more marvellous because Napier had no notations, such as we have now, to guide him. Dr. Glaisher said that he himself always assumed that the geometrical form in which the invention was given to the world was the original form, but Lord Moulton's suggestion, that the original form was arithmetical, was well worth a careful examination.

The second paper was a note sent in by Dr. G. Vacca of Rome, Italy, pointing out that Paciolo's solution, in 1494, of the problem, to find in how many years a sum would double itself at a given rate of compound interest, involved the calculation of the logarithm of 2—the first logarithm computed before Napier.

Professor G. A. Gibson of Glasgow discussed the transition from Napier's original logarithms to those of base 10. He pointed out that the theorem that the sum of the logarithms of two numbers was the logarithm of their product, was not accurately true for Napier's original logarithms; nor would it have been true for Briggs's suggested improvement; it was Napier himself who first suggested making the logarithm of unity zero.

Professor David Eugene Smith of Columbia University gave a paper on the law of exponents in works of the sixteenth century and their influence on Bürgi, whose *Progress-Tabulen* (1620) contained an anti-logarithmic table. The importance of the arithmetical relationships conceived by the Germans approached so closely to the idea of the logarithm that it was cause for wonder that Napier and Briggs were not anticipated. Lieutenant Salik Mourad of the Turkish Navy said that logarithms were brought into Turkey in 1714. Professor Florian Cajori of Colorado College read a paper on algebra in Napier's day and alleged prior inventions of logarithms. A novel point in this paper was to the effect that Benjamin Martin, an English writer of the eighteenth century, claimed that Edward Wright deserved priority over Napier, having published a table of logarithms in 1599, sixteen years before Napier. Wright published in 1599 a "table of latitudes" which could indeed be interpreted as a system of logarithms, but Wright used it simply in correcting certain errors in navigation; the logarithmic concept was at that time foreign to him, and Martin's claim cannot be allowed. Cajori gave a careful analysis of the claims of priority which have been made by different historians for Napier and for Bürgi, and came to the conclusion that most of the statements made in histories and cyclopedias are open to serious objection.

A paper by Dr. Sommerville of St. Andrews, on Napier's rules and trigonometrically equivalent polygons, contained some interesting extensions to non-Euclidean space.

On Monday morning Professor David Eugene Smith occupied the chair and introduced Professor Bauschinger of Strassburg who spoke in German and discussed certain formulas relating to the development of a function of two variables in terms of spherical harmonics. He was followed by Professor H. Andoyer of the University of Paris who spoke in French and explained the construction of his trigonometric and logarithmic tables, published in 1911. These tables contain logarithms to 17 figures for every hundredth of the quadrant. Thereupon, Professor d'Ocagne of the Polytechnic School in Paris gave, in French, two short historical notes; one that the principle of the millionaire calculating machine was invented in 1893 by a young French mechanician, Léon Bollée; the other note had reference to the development of nomography. As there were about twenty papers, altogether, to be presented on Monday morning and the early afternoon, the time of presenting them had necessarily to be greatly curtailed. Among the other speakers of the day were Mrs. Gifford, Dr. J. R. Milne, Mr. Hudson, Mr. H. S. Gay, Mr. J. C. Fergusson, Mr. Schooling, Dr. A. Hutchinson, and Dr. W. F. Sheppard.

The principal papers read at the celebration will be published in a volume and will doubtless prove of great interest to mathematicians. It will contain many matters of historic importance. In connection with the congress there was held a very remarkable exhibition of calculating machines, models, old and recent books on logarithms, and relics of Napier. This display was held beneath the same roof as the recently instituted mathematical laboratory of the University of Edinburgh. Among the Napier relics were sets of "Napier's Bones" and early editions of his books. That there was no tendency to minimize the work of Bürgi as an inventor of logarithms was evident from the fact that Bürgi's *Progress-Tabulen* of 1620 and Gieswald's pamphlet of 1656 in which occurs the first publication of Bürgi's explanation of his tables were among the exhibits. These two publications were loaned by the city of Dantzig. Very few, if any, of the mathematicians present had ever before seen them. Dr. Glaisher who has been interested in logarithmic tables for over a quarter of a century had never before enjoyed the opportunity of examining them. It is proposed, in the forthcoming publication of the proceedings of the celebration, to insert photographs of the title-page and of an inside page of Bürgi's book of 1620.

The exhibit included a superb collection of portable sun-dials which were used before the days of watches, of theodolites, of dialling instruments and of sectors. The exhibition of calculating machines was probably the largest ever seen. It included the highly complicated machines of today, some of which are worked by hand, some by electricity. There was an exhibit of planimeters and integraphs by G. Coradi of Zürich, by A. Ott of Kempten in Bavaria, by the University of Glasgow, and by the University of Edinburgh. There was also a large display of old and new slide rules, periodogram analysis apparatus, chronographs, equation-solvers, synthetizers, coördinographs, logarithmic and semi-logarithmic paper, radian polar paper, and collinear-point nemograms. A conspicuous machine was Edward Roberts's tide-predicting machine, which was awarded the Grand Prix at the Franco-British Exhibition of 1908. A photograph of Lord Kelvin's tide predicting machine was also shown.

Of special interest to geometers and crystallographers was the display of mathematical models. Dr. Sommerville's semi-regular polyhedra and his projection model of the 600-cell in space of four dimensions evoked attention. All sorts of linkages, and anaglyphs producing stereoscopic effects by viewing bi-colored diagrams through absorption screens, were seen in another part of the room.

The Royal Society of Edinburgh published a large and admirable handbook of the exhibition, of 343 pages, edited by E. M. Horsburgh, lecturer on technical mathematics in the Edinburgh University. This book will be a valuable addition to any mathematical library.

THE PARIS REPORT ON CALCULUS IN SECONDARY SCHOOLS.

L'Enseignement for July–September contains the complete reports (referred to in the September MONTHLY) made at the Paris meeting of the International Commission on the Teaching of Mathematics. One of the two subjects there discussed, viz., the results of the introduction of the elementary notions of the calculus into secondary instruction, seems to be of such importance to the readers of the MONTHLY as to justify printing the following translation of the official summary of the general report made by Professor Beke, of Budapest, covering the replies made to the questionnaire. (For greater brevity, the differential and integral calculus will be referred to simply as the calculus.)

"1. *The place of the calculus in the secondary curriculum.*—In all countries where during the last twelve years a new plan of studies for secondary schools has been vigorously begun, a definite place of greater or less importance has been reserved for the notion of function, and as well, with very few exceptions, for the first elements of the calculus.[1]

"*A.* The elements of the infinitesimal calculus figure in the official programs of the schools, or in the plan of study established by the schools themselves, in the following countries: Bavaria, Wurtemberg, Baden, Hamburg, Austria, Denmark, France, Great Britain, Italy, Roumania, Russia, Sweden and Switzerland.

"*B.* They do not figure in the plan of study, but are elective in a large number of schools: Prussia, Saxony, Hungary, Australia; and they probably will be so before long in Holland, Norway, Belgium and Servia.

"2. *The range given to the calculus.*

"(*a*) It is nearly everywhere applied only to functions of one variable.

"(*b*) Instruction is everywhere given in the differentiation of polynomials, of rational functions (or at least quotients of two linear polynomials), and in most countries that of exponential and trigonometric functions and their inverses.

"(*c*) In most countries the notation of Lagrange is preferred to that of Leibnitz.

"(*d*) In most countries the notion of integral or primitive function is introduced. Everywhere the notion of integral follows that of derivative (in Bohemia these are treated simultaneously). In some countries the definite integral precedes the indefinite, but in most the reverse order is followed.

"3. *Applications of the infinitesimal calculus.*

"(*a*) Taylor's series figures in few programs. It is nevertheless taught in schools where the plans of study have for a long while included infinite series; in such cases the series for e^x, a^x, $\sin x$, $\cos x$, $\log (1 + x)$, $(1 + x)^m$, $\arctan x$ are developed. Professor Beke thinks that the treatment of Taylor's series is not yet sufficiently adapted to the secondary school.

"(*b*) The calculus is applied everywhere to investigations of maxima and minima.

"(*c*) It is also applied to physics, at least for the purpose of defining velocity

[1] On the status in America see remarks by Professors Smith and Van Vleck quoted on page 326.

and acceleration, but sometimes it finds a more extended application (centers of gravity, moments of inertia, potential, etc.). In Russia only elementary mathematics is used in physics.

"(*d*) The calculus is applied in geometry to the determination of areas and volumes, and it is here that the new method renders its greatest service on the economical side. But the old methods continue to be applied, especially Cavalieri's principle.

"4. *The question of rigor.*—This is one of the most delicate points. From the side of the higher curriculum it is said that secondary teaching does more harm than good unless it adopts the rigorous method of a scientific treatment; on the other hand, the representatives of secondary instruction assert that the intellectual power of the average pupils does not permit a rigorous treatment of the calculus. Professors in secondary schools need to know the infinitesimal calculus in its modern rigorous form, but in their teaching they need to use an intuitive method, geometrical and mechanical considerations, and to progress gradually toward the necessary abstractions. This is also the surest way to arouse in the pupils the desire for rigor.

"(*a*) Irrational numbers are introduced almost everywhere incidentally in connection with the extraction of roots; the general theory is treated only in exceptional cases.

"(*b*) The notion of limit is introduced everywhere, nowhere do we rely on intuition alone. The elementary theorems relative to limits are adopted almost everywhere without explanation.

"(*c*) No allusion is made to continuous functions which nowhere admit derivatives. In certain schools instructors confine themselves to saying that at certain points the derivative may cease to exist.

"(*d*) In most schools the differential is not introduced; there is a good deal of confusion in the treatment of the notion of differential. It is greatly to be desired that the metaphysical fog of the infinitely small should not enter the secondary teaching.

"5. *Fusion of the calculus with the subject matter of the secondary school.*—The new subjects should not be given as supplementary matter alongside of the old subject matter, but a complete fusion will have to be brought about between the two. The enlargement of the rôle of the notion of function and the introduction of the calculus can be successful only if the old program be reduced and made more economical. There will then result a relief, thanks to the fusion of the new material with the old, and the suppression of old out-of-date subject matter.

"6. *The reform movement and the general opinion of educators.*—The very definite character of the results of our movement may be assured by its success and by the general opinion (l'opinion publique), always alert, of the representatives of education. The movement has gained everywhere the approval of the professors engaged in secondary education, but those engaged in higher education, who look at this from their own special point of view, do not always sympathize with our tendencies.

"We hear the complaint that a course in calculus is not followed with interest

by those who already have some knowledge of it. It is not difficult to refute this assertion. It suffices us to recall the favorable opinions which we have found among university professors in all countries, who look at our movement from a higher point of view."

To this should be added here the following extracts concerning the report of Professor D. E. Smith, the supplementary remarks made by Professor E. B. Van Vleck, and the British report made by Professor C. Godfrey.

" Professor D. E. Smith, the zealous reformer and one of the founders of the Commission, informs us, that the calculus does not figure in the secondary curriculum in the United States; it cannot even be made elective, since the pupils of the upper classes are very much absorbed in the preparation for college entrance examinations. So long as this (the calculus) is not put upon the program of these examinations, there is little chance that it will take a place in the subject matter of secondary teaching. Yet Professor Smith has the hope that before many years the calculus will be introduced in professional (technical) secondary schools. Knowing the great activity shown by our American colleagues, in the past and present, in regard to reform in mathematical teaching (we have only to recall the work of Professors D. E. Smith, Moore and Young) and seeing the immense scope of mathematical activity across the sea which dazzles our eyes and which is assuredly not without a favorable influence on the professors in secondary teaching, finally having confidence in the contact which in spite of distance exists between the workers of the two continents, we cannot doubt that before long the free development of mathematical teaching will have taken the decisive step."

Professor E. B. Van Vleck: " It may be added that to some degree the work of the first year or two of the American college course corresponds, in character, to that of the last year of the German gymnasium and the classes spéciales of the Lycées. The study of calculus is very commonly begun in the second year of the college course, and not infrequently it is taken by students in their first year. Furthermore, graphical representation for simple functions (linear and quadratic functions) has been increasingly introduced as a topic into the algebra of the high schools. From both of these facts it is clear that the tendencies now under discussion at this conference are also manifesting themselves visibly in the United States."

Professor C. Godfrey: " Broadly speaking the movement has received general support in England. Perhaps the most powerful stimulus is that of the engineers, as represented by Professor Perry. The physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigor. The universities have progressively included more calculus in their examination papers for schools; these papers, together with those set by the Civil Service Commissioners (for admission to the army and the public service generally), are the most powerful lever that acts on the school curriculum. . . . Whatever opposition there has been to an introduction of the calculus at an early stage has come from those who fear that a diminished emphasis on the manipulative and formal side of algebra will have a bad effect. The question raised is this: What

algebraic equipment constitutes a firm basis for a superstructure of calculus? This is the only articulate objection that has found a voice. But the main obstacle is that most powerful force in educational matters—*vis inertiæ*."

SUR UN PARADOXE ALGÉBRIQUE APPARENT

par G. LORIA, Université de Gênes, Italie

La contradiction signalée par M. Coolidge aux pages 184–5 de THE AMERICAN MATHEMATICAL MONTHLY de cette anneé peut se faire disparaitre de la manière suivante: Ecrivons l'equation

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0$$

comme il suit

$$-\begin{vmatrix} a & -b & c & -d \\ a' & -b' & c' & -d' \\ b & a & d & c \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Appliquons a présent le théorème de Laplace sur le développement d'un déterminant et nous obtiendrons:

$$(ac' - a'c)^2 + (bd' - b'd)^2 + (ad' - a'd)^2 + (bc' - b'c)^2 - 2(a'b - ab')(c'd - cd') = 0.$$

ou bien, par des transformations algébriques faciles,

$$[(ac' - a'c) - (bd' - b'd)]^2 + [(ad' + bc') - (a'd + b'c)]^2 = 0.$$

Comme $a, b, \ldots$ sont toutes des quantités réelles cette équation unique se décompose dans les deux suivantes,

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c),$$

qui sont précisément celles qu'a trouvees l'éminent professeur de l'Harvard University.

GÊNES, 28, VII, 1914.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Dialogues Concerning Two New Sciences. By GALILEO GALILEI. Translated from the Italian and Latin into English by HENRY CREW and ALFONSO DE SALVIO with an introduction by ANTONIO FAVARO. The Macmillan Company, New York, 1914. xxi+300 pages. $2.00.

Into these dialogues, Galileo gathered, during his last years, practically all of his work that is of value to the physicist or the engineer. The translation into English, at this time, is most desirable as the few copies of the two former translations, by Thomas Salusbury in 1665 and Thomas Weston in 1730, are quite inaccessible and many passages are either ambiguous or unintelligible to the modern reader.

The translation strictly follows the text of the National Edition of the Works of Galileo recently completed by Antonio Favaro. The text is essentially the same as that of the original Elzevir edition of 1638. All comments and annotations have been omitted with the exception of a few footnotes properly labeled (Trans.) in order to preserve the original as nearly as possible, and the translation is as literal as is consistent with clearness and modernity. Where there is any deviation from this rule and in case of some technical terms the original Italian or Latin phrase is added enclosed in brackets. The original illustrations are reproduced.

The presentation of the science in the form of dialogues is exceedingly striking and effective. The dialogues continue during four successive days. In the first day the three men discuss resistance which solid bodies offer to fracture, friction, cohesion, the finite and the infinite, laws of falling bodies, terminal velocity, the density of air, isochronous vibrations of the pendulum, resonance, etc. During the second day the same men discuss levers, beams, cohesion, resistance to fracture, and strength of hollow tubes.

On the third day they begin the discussion of the "second new science," treating of motion. They consider the laws of uniform motion and of uniformly accelerated motion, and the time of descent of a body down an inclined plane under various conditions.

The fourth day is taken up with the consideration of the composition of a horizontal uniform motion with a vertical uniformly accelerated motion, the velocity of a projectile at any point of its path, the maximum range and various other problems concerning projectiles, the sagging of a stretched cord. The term momentum is used and impulsive forces are mentioned but not explained.

It will repay anyone, who has not already done so, to read this admirable book of historic as well as intrinsic interest.

ANTHONY ZELENY.

UNIVERSITY OF MINNESOTA.

Plane Trigonometry and Applications. By E. J. WILCZYNSKI, Professor of Mathematics in the University of Chicago. Edited by PROFESSOR H. E. SLAUGHT. Allyn and Bacon, 1914. xi+265 pages. $1.25.

Logarithmic and Trigonometric Tables. By the same author and editor. Allyn and Bacon, 1914. xx+97 pages. $1.00.

Every friend of mathematics and every friend of mathematical teaching must rejoice when scholars of national and international reputation give their time to the compiling of text-books in the more elementary branches of the science. It is a serious mistake to suppose that high school teachers are best equipped to write high school texts. It is particularly fortunate when the author is not only a scholar of wide outlook, but also a man with a keen instinct for teaching. When, moreover, the work of such a man has passed under the searching eye of an editor who is also an acknowledged leader in mathematical teaching, it is natural to expect much in advance of the book.

The work is divided into two parts, the first of which is devoted to the solution of triangles. The trigonometric functions of acute angles are defined as usual, and the notion of the general angle is deferred till part two. This is precisely the method which a less experienced teacher would avoid. Why not give the general definition at the very first, and avoid the readjusting of ideas made necessary by two different definitions? This is, indeed, a very vital and important matter in pedagogy. There is a tendency of late years to hurry the child into algebra before it has had time to get acquainted with the simpler conceptions of arithmetic. Why study arithmetic when you might as well be studying the more general science? Why not, indeed? And why not study the theory of groups in the grades, seeing that algebra is, in the last analysis, a chapter in abstract groups? The reviewer looks with admiration on the skilful means by which all of the formulæ for the solution of triangles, both right and oblique, are developed without the assistance of coördinate geometry, and even without the use of the addition theorems. Even the law of tangents has been derived by purely synthetic methods. Professor Wilczynski has hit upon the happy device of giving a central position to the area problem. All the formulæ of trigonometry present themselves naturally with a unifying point of attack. This part is also published separately and would seem to be admirably adapted for use in secondary schools.

Much attention is given in Part One to the subject of computation. The author is a computor of many years experience and can speak with authority on this subject. The *negligence* of expressing a result in fewer decimal places than one is able to guarantee, and the *dishonesty* of expressing it in more are topics dwelt on with some warmth. The reviewer is a little disconcerted, however, to find the so-called *abbreviated method* of multiplication referred to as preferable to the ordinary method. They are both so very bad that it is difficult to say which is worse.

There is also in this chapter a lucid discussion of logarithms, which with the illustrative examples will be easily grasped by any class. A short description is also given of the slide-rule.

The book of tables published with the trigonometry should be noted in connection with part one. It is a very attractively constructed table of five-place logarithms, both of numbers and of trigonometric functions. The spacing is generous and well calculated to relieve the eye of undue strain. A number of smaller tables are also given in the book, such as extra tables for " small " angles; a table of four places for the natural functions; a table of squares; a table for transforming angles; a list of constants; and three small three-place logarithmic tables. Altogether a very useful book of tables.

Part Two of the trigonometry contains, besides the discussion of the general angle, and the addition theorems and consequences of them, certain chapters not usually met with in treatises on the elements of this subject. The chapter on directed line-segments is a good introduction for the study of projective geometry and for the study of vector analysis. The chapter on inverse functions is one that will be of value to every student who will go on to the study of the calculus. It is a refreshing thing to see an American text-book with the notation " Arc sine."

Perhaps the most unusual feature of the book is the insertion of the chapter on the Theory of Wave Motion which closes the book. It is a subject of unusual interest and importance, and the chapter is one of the most beautiful in the book. We doubt, however, whether many teachers will venture to include it in a course where students are apt to have as little mathematical maturity as most students of trigonometry do. Such things, however, are worth while placing where teachers and students can have a look at them sometimes; and who knows whether twenty years from now they may be included in the earlier years of the student's life just as now we include the study of the calculus in the freshman and sophomore years and not in the very last year as was the custom twenty years ago.

Much of interest in historical matter has been introduced " not in the form of detached historical notes, but organically connected with the topic under discussion." This appears to the reviewer to be a decided improvement on the usual method, but he cherishes the belief that the effect would be much better if the historical matter were collected into a separate chapter at the end of the book, and the subject discussed as a whole,—but each one to his taste in such matters.

The reviewer has not given the book a microscopic examination for printer's errors, and mis-spelled words. He has not even worked out any of the trigonometric identities, of which he is glad to note the fewness, but he has been impressed with the inviting appearance of the pages, and the successful arrangement of the matter on the page. These details count for much more in making a usable book than teachers sometimes think.

D. N. LEHMER.

UNIVERSITY OF CALIFORNIA.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up no solution of 420 had been received.

424. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{n}{a}-\binom{n-1}{a}\binom{i}{1}+\binom{n-2}{a}\binom{i}{2}-\binom{n-3}{a}\binom{i}{3}+\cdots+(-1)^i\binom{n-i}{a},$$

and consider the cases $i = a$ and $i > a$.

425. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Solve for x and y the equations: $2^{x+y} = 6$ and $2^{x+1} = 3^y$.

GEOMETRY.

When this issue was made up solutions had been received for numbers 370, 430, 432–3, 447–8, 450–1. Please give attention to 427, 442, 446, 449.

452. Proposed by NATHAN ALTSHILLER, University of Washington.

Through a given point a secant is drawn that meets three given concurrent lines in the points A, B, C respectively. Determine the position of the secant by the condition $AB/BC = K$, K being given.

453. Proposed by CLIFFORD N. MILLS, South Dakota Agricultural College.

Prove geometrically the formulæ for $\sin 2\beta$, $\cos 2\beta$, $\sin 3\beta$, $\cos 3\beta$.

454. Proposed by LOUIS ROUILLION, Mechanics Institute, New York City.

Show how to construct an equilateral triangle with its vertices lying on three lines not equally spaced.

CALCULUS.

When this issue was made up solutions had been received for numbers 358–9, 361–2, 364, 371, 373. Please give attention to 332, 339, 340, 342, 348, 353, 360, 363.

374. Proposed by C. N. SCHMALL, New York City.

Show that, on a *Mercator's Chart*, a great circle of a sphere whose radius is r will be represented by a curve whose equation is of the form

$$c(e^{y/r}-e^{-(y/r)}) = 2\sin\left(\frac{x}{r}+\theta\right).$$

(*Note.* See EISENHART's *Differential Geometry*, § 46, pp. 107–108; OSGOOD's *Calculus*, pp. 331–333, § 6.)

375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation,

$$x^2(a-bx)\frac{d^2y}{dx^2}-2x(2a-bx)\frac{dy}{dx}+2(3a-bx)y = 6a^2.$$

MECHANICS.

When this issue was made up solutions had been received for numbers 292, 293, 295–6–7–8–9. Please give attention to 272, 277–8–9, 286–7, 290–1, 294, 300.

301. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

A wire is hanging from two points in the same horizontal plane. If the difference between the length of the wire and the actual distance between the supports is very small, show that

$$s = x\left(1 + \frac{x^2}{6c^2}\right),$$

where s is one half the length of the wire, c is the horizontal tension at the lowest point divided by w the load per unit of horizontal distance, and x is the distance of lowest point of the curve to the point of support.

NUMBER THEORY.

When this issue was made up solutions had been received for numbers 212–13, 215–16, 218, 220, 223. Please give attention to 191–2, 196, 198, 201–2, 205, 208–9–10–11, 214, 217, 219, 221–2.

224. Proposed by PATRICK WALSH, New Orleans, Louisiana.

Find the sides, in rational numbers, of a right angled triangle whose area is $5\frac{1}{2}$.

225. Proposed by W. DE W. CAIRNS, Oberlin College.

L'Intermédiaire for June, 1914, contains the following problem:

"If we write the terms of the arithmetic series 1, 5, 9, 13, 17, 21, 25, 29, 33, ··· as follows:

$$\begin{array}{ccccccc} 1 & & & & & & \\ 5 & 9 & 13 & & & & \\ 17 & 21 & 25 & 29 & 33 & & \\ 37 & 41 & 45 & 49 & 53 & 57 & 61 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array},$$

it is seen that the sum of the terms of each line is a cube, and that these are the cubes of the successive "odd integers. How is this shown?"

It is here proposed not only to prove this, but to generalize the theorem as suggested, using, however, the simpler (and better known) case which includes all of the successive integers:

$$\begin{array}{cccc} 1 & & & \\ 3 & 5 & & \\ 7 & 9 & 11 & \\ 13 & 15 & 17 & 19 \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

412. Proposed by H. L. SLOBIN, University of Minnesota.

Form the algebraic equation whose roots are:

$$a_1 = \cos\frac{\pi}{9}, \qquad a_2 = -\cos\frac{2\pi}{9}, \qquad a_3 = -\cos\frac{4\pi}{9}.$$

SOLUTION BY CLIFFORD N. MILLS, Brookings, South Dakota.

From the suggestions given by the proposer,

$$\cos r\pi = \frac{(-1)^r + (-1)^{-r}}{2}.$$

Hence,

$$a_1 = \frac{(-1)^{1/9}+(-1)^{-1/9}}{2},\quad a_2 = -\frac{(-1)^{2/9}+(-1)^{-2/9}}{2},\quad a_3 = -\frac{(-1)^{4/9}+(-1)^{-4/9}}{2},$$

and the equation to be constructed is

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0.$$

We have

$$\Sigma a_1 = \frac{-(-1)^{8/9} - (-1)^{6/9} + (-1)^{5/9} + (-1)^{3/9} - (-1)^{2/9} - 1}{2(-1)^{4/9}},$$

which involves the nine 9th roots of (-1).

From $x^9 + 1 = 0$ we have

$$x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0,$$

which contains all the *complex* 9th roots of (-1).

Hence

$$x^8 + x^6 - x^5 - x^3 + x^2 + 1 = x^7 - x^4 + x,$$

and this is equal to zero since

$$x^7 - x^4 + x = x(x^6 - x^3 + 1),$$

in which the factor $x^6 - x^3 + 1 = 0$.

Therefore

$$\Sigma a_1 = \frac{-(-1)^{1/9}[(-1)^{6/9} - (-1)^{3/9} + 1]}{2(-1)^{4/9}} = 0.$$

Again,

$$a_1a_2 + a_1a_3 + a_2a_3 = -\frac{1}{4}\left[\frac{(-1)^{6/9} + (-1)^{4/9} + (-1)^{2/9} + 1}{(-1)^{3/9}}\right.$$

$$\left. + \frac{(-1)^{10/9} + (-1)^{8/9} + (-1)^{3/9} + 1}{(-1)^{5/9}} - \frac{(-1)^{12/9} + (-1)^{8/9} + (-1)^{4/9} + 1}{(-1)^{6/9}}\right].$$

Hence

$$\Sigma a_1a_2 = \frac{1}{4}\frac{(-1)^{12/9} - (-1)^{11/9} - 2(-1)^{9/9} + (-1)^{8/9} - (-1)^{7/9} - (-1)^{6/9} + (-1)^{4/9} - 2(-1)^{3/9} - (-1)^{1/9} + 1}{(-1)^{6/9}}.$$

Now

$$x^8 - x^7 - x^5 + x^4 - x^3 - x + 1 = -x^6 - x^3.$$

Hence

$$\Sigma a_1a_2 = \frac{1}{4}\frac{x^9 - 2x^6 - x^3 + 1}{x^3} = \frac{1}{4}\frac{x^9 - 3x^3 + 1}{x^3} = -\frac{3}{4},$$

making use of $x^9 + 1 = 0$ and $x^6 = x^3 - 1$.

Finally,

$$a_1a_2a_3 = \frac{(-1)^{14/9} + (-1)^{13/9} + (-1)^{10/9} + (-1)^{8/9} + (-1)^{6/9} + (-1)^{4/9} + (-1)^{2/9} + 1}{8(-1)^{7/9}}$$

$$= \frac{-(-1)^{5/9} - (-1)^{3/9} - (-1)^{1/9} + (-1)^{7/9} + (-1)^{5/9} + (-1)^{3/9} + (-1)^{1/9}}{8(-1)^{7/9}}$$

$$= \frac{(-1)^{7/9}}{8(-1)^{7/9}} = \frac{1}{8},$$

making use of $x^8 + x^6 + x^4 + x^2 + 1 = x^7 + x^5 + x^3 + x$.

Hence, the required equation is

$$x^3 - \tfrac{3}{4}x - \tfrac{1}{8} = 0.$$

Solved similarly by H. C. FEEMSTER, and also by C. E. GITHENS by merely substituting the values of a_1, a_2, a_3 and reducing by means of trigonometric relations. Mr. Feemster also sent in such a second solution.

Editorial note. The details of the work on this problem may be simplified by use of Gauss' periods. See WEBER's *Algebra*, volume I, chapter XVI.

413. Proposed by C. N. SCHMALL, New York City.

Apply Euler's transformation to show that

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \frac{1+x}{(1-x)^3}.$$

SOLUTION BY W. C. BRENKE, University of Nebraska.

This is a special case of the series

$$S(x) = 1 + 2^rx + 3^rx^2 + 4^rx^3 + \cdots; \; r = \text{positive integer}.$$

Such series may be summed by repeated use of Euler's transformation based on the factor $(1 - x)$. The whole operation may be performed at once as follows, the process being valid for $|x| < 1$.

Multiply both sides of the last equation by

$$(1-x)^{r+1} = 1 - {}_{r+1}C_1x + {}_{r+2}C_2x^2 - \cdots + (-1)^{r+1}x^{r+1}.$$

Arranging the right member in powers of x, we find that the coefficients of x^{r+1}, x^{r+2}, $\cdots$ are all zero by virtue of the formula (special case of (3) p. 183 of Chrystal's Algebra, Vol. 2)

$$n^r - (n-1)^r{}_nC_1 + (n-2)^r{}_nC_2 - \cdots + (-1)^{n-1}{}_nC_{n-1} = 0; \; r < n.$$

Hence

$$(1-x)^{r+1}S(x) = 1 + (2^r - {}_{r+1}C_1)x + (3^r - 2^r{}_{r+1}C_1 + {}_{r+1}C_2)x^2 + \cdots + (r^r - (r-1)^r{}_{r+1}C_1 + \cdots + (-1)^r{}_{r+1}C_r)x^r.$$

The result may also be written in the form

$$S(x) = \frac{S_r(x) - {}_{r+1}C_1xS_{r-1}(x) + {}_{r+1}C_2x^2S_{r-2}(x) + \cdots + (-1)^r{}_{r+1}C_rx^r}{(1-x)^{r+1}},$$

where

$$S_n(x) = 1 + 2^rx + 3^rx^2 + \cdots + (n+1)^rx^n.$$

When $r = 2$ this gives the sum of the proposed series.

Also solved by the PROPOSER.

414. Proposed by R. D. CARMICHAEL, Indiana University.

Prove by means of an example that one of the series

$$\sum_{k=1}^{\infty}\frac{1}{c_k}, \quad \sum_{k=1}^{\infty}\frac{1}{c_k - 1}, \qquad c_k \neq 0, 1,$$

may be divergent while the other is convergent.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

The following two series fit the conditions of the problem:

$$\frac{1}{\sqrt{2}}+\frac{1}{-\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{-\sqrt{5}}+\cdots,$$

$$\frac{1}{\sqrt{2}-1}+\frac{1}{-\sqrt{3}-1}+\frac{1}{\sqrt{4}-1}+\frac{1}{-\sqrt{5}-1}+\cdots.$$

The first converges because it is an alternating series whose terms steadily decrease in absolute value and approach zero. That the second is divergent may be seen by bracketing the terms in pairs. For, if n is even, we have

$$\frac{1}{\sqrt{n}-1}+\frac{1}{-\sqrt{n+1}-1}=\frac{-\sqrt{n}+\sqrt{n+1}+2}{(\sqrt{n}-1)(\sqrt{n+1}-1)}>\frac{2}{n+1};$$

whence the series is divergent as one sees by comparison with the divergent series

$$\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\cdots.$$

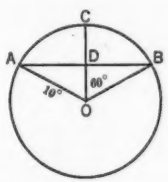
GEOMETRY.

440. Proposed by W. L. WATSON, Moundsville, West Va.

A solid sector is cut out of a sphere 10 feet in radius by a cone whose vertical angle is 120°. Find the radius of the sphere whose volume is equal to that of the sector.

SOLUTION BY EMMA M. GIBSON, Drury College.

The volume of the solid sector O–ABC is equal to that of the cone AOB, having a vertex angle of 120° and altitude OD equal to $10\cos 60°$, plus the volume of a spherical segment of altitude $CD = 10 - 10\cos 60° = 5$, and diameter of base $AB = 2 \times 10 \sin 60° = 10\sqrt{3}$. Then



$$\text{Volume of cone} = \frac{1}{3}\text{base} \times \text{altitude} = \frac{1}{3}\pi 75 \times 5 = 125\pi.$$

$$\text{Volume of segment} = \frac{1}{3}\pi \times 25 \times 25 = \frac{625}{3}\pi.$$

$$\therefore \text{Volume of solid sector} = 125\pi + \frac{625\pi}{3} = \frac{1000\pi}{3}.$$

Let r' be the radius of sphere whose volume is equal to the volume of the solid sector. Then $\frac{4}{3}\pi r'^3 = \frac{1000}{3}\pi$.

$$\therefore\ r' = \sqrt[3]{250} = 5\sqrt[3]{2}.$$

Also solved by A. M. HARDING, PAUL CAPRON, WALTER C. EELLS, ELMER SCHUYLER, H. C. FEEMSTER, HORACE OLSON, C. E. GITHENS, GEO. W. HARTWELL, and CLIFFORD N. MILLS.

CALCULUS.

354. Proposed by C. N. SCHMALL, New York City.

Prove

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

SOLUTION BY A. G. CARIS, Defiance, O.

(The outline for this proof is taken from notes of a course in *Definite Integrals* given by Professor G. A. Bliss at the University of Chicago.)

The following preliminary theorems are used in the proof.

(1) $$\sin z = z\prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{\nu^2\pi^2}\right).$$

(2) $$\cos z = \prod_{\nu=1}^{\infty}\left(1 - \frac{4z^2}{(2\nu-1)^2\pi^2}\right).$$

(3) $$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

(4) $$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

(5) $$\Gamma(1+a) = a\Gamma(a).$$

(6) $$y^{-a}\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-yx}dx.$$

PROOF OF THEOREM (1).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2} = 2\sin\frac{z}{2}\sin\frac{z+\pi}{2}.$$

By repeated applications of this formula we obtain

$$\sin z = 2^{2^n-1}\sin\frac{z}{2^n}\cdot\sin\frac{z+\pi}{2^n}\cdot\sin\frac{z+2\pi}{2^n}\cdots\sin\frac{z+(2^n-1)\pi}{2^n}.$$

This may be written more briefly by substituting $2^n = p$.

$$\sin z = 2^{p-1}\prod_{\nu=0}^{p-1}\sin\frac{z+\nu\pi}{p}.$$

Combine these factors in pairs, taking together such factors that the sum of the coefficients of π shall be p. The general form of these products will be

$$\sin\frac{z+\nu\pi}{p}\sin\frac{z+(p-\nu)\pi}{p}.$$

By trigonometric transformation, this becomes

$$\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}.$$

There will be $\left(\frac{p}{2}-1\right)$ of these factors, the factor $\sin\frac{z}{p}$, and the factor $\sin\frac{z+\frac{p}{2}\pi}{p}$ which equals $\cos\frac{z}{p}$.

Whence,
$$\sin z = 2^{p-1}\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right). \qquad (A)$$

Dividing both sides of this equation by z,

$$\frac{\sin z}{z}=\frac{2^{p-1}}{p}\cdot\frac{\sin\frac{z}{p}}{\frac{z}{p}}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right).$$

Passing to the limit of both sides as $z \doteq 0$, we have

$$1=\frac{2^{p-1}}{p}\prod_{\nu=1}^{(p/2)-1}\sin^2\frac{\nu\pi}{p}.$$

Dividing both members of (A) by this result, we have

$$\sin z = p\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(1-\frac{\sin^2\frac{z}{p}}{\sin^2\frac{\nu\pi}{p}}\right).$$

Taking the limit of this expression as $p \doteq \infty$, we have

$$\sin z = z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\pi^2}\right).$$

PROOF OF THEOREM (2).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2}=z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\cdot\pi^2}\right).$$

From (1),

$$\sin\frac{z}{2}=\frac{z}{2}\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{(2\nu)^2\pi^2}\right).$$

Dividing the first of these two equations by the second, we have

$$2\cos\frac{z}{2} = 2\prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{(2\nu-1)^2\pi^2}\right).$$

From this we have at once

$$\cos z = \prod_{\nu=1}^{\infty}\left(1 - \frac{4z^2}{(2\nu-1)^2\pi^2}\right).$$

PROOF OF THEOREM (3).

From (1) and (2) we have

$$\sin \pi a = \pi a \prod_{\nu=1}^{\infty}\left(1 - \frac{a^2}{\nu^2}\right), \qquad \cos \pi a = \prod_{\nu=1}^{\infty}\left(1 - \frac{4a^2}{(2\nu-1)^2}\right).$$

At once we have

$$\log \sin \pi a = \log \pi + \log a + \log\left(1 - \frac{a^2}{1^2}\right) + \log\left(1 - \frac{a^2}{2^2}\right) + \cdots$$

and

$$\log \cos \pi a = \log\left(1 - \frac{4a^2}{1^2}\right) + \log\left(1 - \frac{4a^2}{3^2}\right) + \log\left(1 - \frac{4a^2}{5^2}\right) + \cdots.$$

By differentiation of these two series, we have

$$\pi \cot \pi a = \frac{1}{a} - \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} - \frac{2a}{3^2 - a^2} - \cdots$$

$$\pi \tan \pi a = \frac{8a}{1^2 - 4a^2} + \frac{8a}{3^2 - 4a^2} + \frac{8a}{5^2 - 4a^2} + \cdots.$$

Substituting $a/2$ for a, we have

$$\pi \tan \frac{\pi a}{2} = \frac{4a}{1^2 - a^2} + \frac{4a}{3^2 - a^2} + \frac{4a}{5^2 - a^2} + \cdots.$$

$$\pi \tan \frac{\pi a}{2} + \pi \cot \pi a = \pi\left[\frac{\sin \frac{\pi a}{2}}{\cos \frac{\pi a}{2}} + \frac{\cos^2 \frac{\pi a}{2} - \sin^2 \frac{\pi a}{2}}{2 \sin \frac{\pi a}{2} \cos \frac{\pi a}{2}}\right] = \frac{\pi}{\sin \pi a}.$$

Adding the corresponding series, we have

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

PROOF OF THEOREM (4).

$$\int_0^a \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \frac{x^{a-1}}{1+x}\,dx + \int_1^a \frac{x^{a-1}}{1+x}\,dx.$$

Transforming the second integral by the substitution, $x = \frac{1}{y}$,

$$\int_1^a \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \frac{y^{-a}}{1+y}\,dy,$$

$$\int_0^1 \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \{x^{a-1} - x^a + x^{a+1} - x^{a+2} + \cdots\}\,dx,$$

$$\int_0^1 \frac{y^{-a}}{1+y}\,dy = \int_0^1 \{y^{-a} - y^{-a+1} + y^{-a+2} - \cdots\}\,dy.$$

Integrating these two series term by term, we have,

$$\int_0^a \frac{x^{a-1}}{1+x}\,dx = \left\{\frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \cdots\right\} + \left\{\frac{1}{1-a} - \frac{1}{2-a} + \frac{1}{3-a} - \cdots\right\}$$

$$= \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

But this, by (3), is $\frac{\pi}{\sin \pi a}$.

$$\therefore \quad \int_0^a \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

Proofs of theorems (5) and (6) may be found in almost any elementary discussion of the properties of $\Gamma(a)$, and need not be given here.

We are now to prove

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

By (5), $\Gamma(1+a) = a\Gamma(a)$.

Our problem is thus reduced to the proof of

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}.$$

By (6),
$$\frac{\Gamma(a)}{y^a} = \int_0^a x^{a-1} e^{-yx}\,dx.$$

Multiplying both sides by e^{-y}, we have

$$\frac{e^{-y}\Gamma(a)}{y^a} = \int_0^a x^{a-1} e^{-y(x+1)}\,dx.$$

Taking the integral of both sides with respect to y from 0 to a, we have

$$\Gamma(a) \int_0^a \frac{e^{-y}}{y^a}\,dy = \int_0^a \int_0^a x^{a-1} e^{-y(x+1)}\,dx\,dy = \int_0^a x^{a-1} \int_0^a e^{-y(x+1)}\,dy\,dx$$

$$= \int_0^a \frac{x^{a-1}}{1+x}\,dx.$$

By (4),

$$\int_0^a \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a}.$$

$$\int_0^a \frac{e^{-y}}{y^a}dy = \int_0^a y^{-a}e^{-y}dy = \int_0^a y^{(1-a)-1}e^{-y}dy = \Gamma(1-a).$$

$$\therefore \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a},$$

and

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

Also solved by T. M. BLAKSLEE, C. N. SCHMALL, A. M. HARDING, A. L. MCCARTY, and J. W. CLAWSON.

355. Proposed by C. N. SCHMALL, New York City.

Given the curve of the nth degree,

$$y^n - (a+bx)y^{n-1} + (c+dx+ex^2)y^{n-2} + \cdots = 0,$$

show that if each ordinate is divided by the corresponding subtangent, the sum of all the resulting ratios will be a constant.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

The question should read: "show that if for a given abscissa each ordinate . . . ."

The sum of all the ordinates corresponding to a given abscissa x_1 is equal to minus the coefficient of y^{n-1}, viz., $+ (a + bx_1)$.

Hence, the sum of the derivatives of the several ordinates will be $+ b$.

But each subtangent is the ordinate divided by the slope of the curve at the top of the ordinate. Hence any ordinate divided by its corresponding subtangent is equal to the slope of the curve at the top of that ordinate. We have just shown that the sum of these slopes for all the ordinates corresponding to a given abscissa is a constant. This proves the problem, as amended. See EDWARDS' *Differential Calculus*, page 151.

Also solved by the PROPOSER.

MECHANICS.

274. Proposed by G. B. M. ZERR.

A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid. If $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then $l = (a-b)4b/a$, $n = a/(a-2b)$, where a = radius of fixed circle, b = radius of rolling circle.

REMARK BY A. H. WILSON, Haverford College.

The statement of this problem is incomplete. As it stands it is not a problem at all.

275. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, England.

If a particle be attracted towards the angular points of a regular hexagon by forces equal to r^{-k}, at distance r, find the condition for stability of equilibrium.

SOLUTION BY A. H. WILSON, Haverford College.

The discussion is for the center of the hexagon, an obvious position of equilibrium. Let this point be taken as origin, the hexagon placed with two vertices on the x-axis. Let r_0 be the side of the hexagon.

Form the potential function V. For one attracting center this is of the form $\int_{r_0}^{r} r^{-h}dV = r^{1-h}/(1-h) - r_0^{1-h}/(1-h)$. (The value $h = 1$ must be excepted.) For simplicity the origin is taken as a standard position.

$$V = \Sigma r^{1-h}/(1-h) - V_0.$$

Each r is of the form $[(x-s)^2 + (y-t)^2]^{1/2}$. Expand V in the neighborhood of the origin in powers of x and y. For r^{1-h} this expansion is

$$r^{1-h} = r_0^{1-h} + s(1-h)r_0^{-h-1}x + t(1-h)r_0^{-h-1}y + \tfrac{1}{2}[(1-h)r_0^{-h-1} - (1-h^2)s^2r_0^{-h-3}]x^2$$
$$-(1-h^2)r_0^{-h-3}st\cdot xy + \tfrac{1}{2}[(1-h)r_0^{-h-1} - (1-h^2)r_0^{-h-3}t^2]y^2 + \cdots.$$

For (s, t) substitute successively the coördinates of the hexagon points and sum:

$$V = V_0 - \tfrac{1}{2}(2+3h)r_0^{-h-1}(x^2+y^2) + \cdots.$$

In the neighborhood of the origin then the equations of motion are

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = (2+3h)r_0^{-h-1}x,$$

$$m\frac{d^2y}{dt^2} = -\frac{\partial V}{\partial y} = (2+3h)r_0^{-h-1}y.$$

For stable equilibrium we must have here

$$2 + 3h < 0.$$

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

NEW QUESTIONS.

19. How many known proofs are there of the proposition that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides? Where are the proofs to be found?

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

21. For the diophantine equation

$$x^3 - y^3 = 17$$

there are known the following solutions;

$$x = \quad 3, \quad 4, \quad 5, \quad 9, \quad 23, \quad 282, \quad 375, \quad 378661,$$
$$y = -2, \quad -1, \quad 2, \quad 4, \quad 5, \quad 43, \quad 52, \quad 5234.$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

REPLY.

14. In the process of solving a certain physical problem Professor H. S. Uhler, of Yale University, was led to the definite integral

$$\int_0^a (a^2 - x^2)x\,dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y}\,dy,$$

for which he found the value

$$\frac{1}{c^3}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right],$$

a and c being positive constants. Professor Uhler would like to see how other persons attack the problem of evaluating this integral.

I. Remarks by A. M. Harding, University of Arkansas.

Let

$$u = \int_0^a (a^2 - x^2)xf(x)dx,$$

where

$$f(x) = \int_{a-x}^{a+x} \frac{e^{-cy}}{y}\,dy.$$

Integrating by parts we have

$$u = -\frac{(a^2 - x^2)^2}{4} f(x)\Big|_0^a + \frac{1}{4}\int_0^a (a^2 - x^2)^2 f'(x)dx;$$

or,

$$u = \frac{1}{4}\int_0^a (a^2 - x^2)^2 f'(x)dx.$$

Now,

$$f'(x) = \frac{e^{-c(a+x)}}{a+x} + \frac{e^{-c(a-x)}}{a-x} = \frac{e^{-ac}}{a^2 - x^2}[a(e^{cx} + e^{-cx}) + x(e^{cx} - e^{-cx})].$$

Therefore

$$u = \frac{ae^{-ac}}{4}\int_0^a (a^2 - x^2)(e^{cx} + e^{-cx})dx + \frac{e^{-ac}}{4}\int_0^a (a^2x - x^3)(e^{cx} - e^{-cx})dx$$
$$= \frac{ae^{-ac}}{4}u_1 + \frac{e^{-ac}}{4}u_2, \text{ say.}$$

Repeated integration by parts gives

$$u_1 = \frac{a^2 - x^2}{c}(e^{cx} - e^{-cx})\Big|_0^a + \frac{2}{c}\int_0^a x(e^{cx} - e^{-cx})dx$$
$$= \frac{2}{c^2}\left[a(e^{ac} + e^{-ac}) - \frac{1}{c}(e^{ac} - e^{-ac})\right].$$

$$u_2 = \frac{a^2x - x^3}{c}(e^{cx} + e^{-cx})\Big|_0^a - \frac{1}{c}\int_0^a (a^2 - 3x^2)(e^{cx} + e^{-cx})dx$$

$$= -\frac{1}{c^2}\left[(a^2 - 3x^2)(e^{cx} - e^{-cx})\Big|_0^a + 6\int_0^a x(e^{cx} - e^{-cx})dx \right]$$

$$= -\frac{1}{c^2}\left[-2a^2(e^{ac} - e^{-ac}) + \frac{6}{c}\left\{ a(e^{ac} + e^{-ac}) - \frac{1}{c}(e^{ac} - e^{-ac}) \right\} \right]$$

$$= \frac{1}{c^2}\left[2a^2(e^{ac} - e^{-ac}) - \frac{6a}{c}(e^{ac} + e^{-ac}) + \frac{6}{c^2}(e^{ac} - e^{-ac}) \right].$$

Substituting the values of u_1 and u_2 in the expression above for u we obtain the desired result.

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

"The propagation of electric waves in wireless telegraphy" is the title of a paper by Professor George R. Dean, of the Missouri School of Mines, in *The Electrician* for September 11, 1914.

Dr. L. L. DINES, formerly professor of mathematics at the University of Arizona, has been elected to an associate professorship at the University of Saskatchewan.

The Southwestern Section of the American Mathematical Society held its eighth regular meeting at the University of Nebraska on Saturday, November 28, 1914.

Mr. J. L. RILEY, formerly instructor in mathematics at the University of Oklahoma, has accepted a fellowship in Rice Institute, Houston, Texas, for the year 1914–15. Professor E. P. R. DUVAL has returned to his former position as associate professor of mathematics at the University of Oklahoma.

Dr. E. E. Whitford has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Mr. Joel D. Eshleman, A.B., has been appointed instructor in mathematics in Adelbert College, Western Reserve University.

Dr. E. B. Stouffer, formerly instructor in mathematics in the University of Illinois, has accepted an assistant professorship in the University of Kansas.

Professor M. Frechet, University of Poitiers, France, who was to lecture at the University of Illinois during the present year, has joined the French army,

The Cambridge University Press of England has published "Tables for Statisticians and Biometricians" by Karl Pearson in quarto form, 228 pages.

The University of Chicago Press announces as in press "The Theory of Collineation Groups" by Professor H. F. Blichfeldt, of Leland Stanford Junior University.

A second revised edition of the English translation by Dr. G. G. Morrice of Professor Felix Klein's well-known "Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree" has just been issued from the press of Kegan Paul, Paris; Trubner and Co., London.

The majority of the American reports to the International Commission on the Teaching of Mathematics are no longer given free distribution by the U. S. Bureau of Education; it is therefore necessary to correct the statement made at the end of the September MONTHLY. On application, however, to the Commissioner of Education, Washington, D. C., a bulletin of the publications of the Bureau will be sent free, giving the prices at which these may be obtained. The whole set of reports above mentioned costs less than one dollar.

At the University of Chicago the departments of mathematics and mathematical astronomy combine as one body, on the one hand in the biweekly club meeting for reports of research in progress, and on the other hand in certain stated meetings for social intercourse. Under the latter head, a general dinner is held in the autumn for the advanced students and members of the staff, a dinner in the spring for the fellows of the departments provided by the members of the staff, and a general dinner in the summer for advanced students and staff. The members of the staff also meet regularly each month for a dinner and evening discussion of matters pertaining to the departments. The autumn general dinner was held on Wednesday evening, October 14, 1914, with an attendance of about fifty.

In recent numbers of *Comptes Rendus* of the French Academy of Science appear the following preliminary notes: May 25, Professor T. H. Gronwall on "The series of Laplace"; June 8, Professor T. H. Gronwall on "Some methods of summation and their application to Fourier's series"; June 15, Professor C. N. Moore on "The relation between certain methods for the summation of a divergent series." The last two are independent announcements to the effect that the method of de La Vallée-Poussin is more general than that of Cesàro.

Among recent publications of interest to teachers of mathematics is "The Teaching of Mathematics" by Professor R. E. Manchester, of the State Normal School, Oshkosh, Wisconsin. This text of 75 pages, published by C. W. Bardeen, Syracuse, after treating the practical benefits of the study of mathematics and the leading considerations in the teaching of arithmetic, algebra, and geometry, deals with the place of mathematics in vocational schools and with methods and modes of presenting these subjects.

The Central Association of Science and Mathematics Teachers met in the Hyde Park High School, Chicago, November 27 and 28. According to the program as issued by the secretary, Mr. W. L. Eikenberry, University High School, University of Chicago, the subject of last year "Vocational courses in science and mathematics" was continued as a part of this year's subject for discussion, viz., "Problems arising in adjusting applied science to high school curricula." A full report of report of the meetings of the Mathematics Section will be made in a later issue.

The Macmillan Co. has recently published "The theory of relativity," a book of 295 pages by L. Silberstein, lecturer in natural philosophy at the University of Rome. It is based, in part, upon a course of lectures delivered in University College, London, in 1912–13.

The British Board of Education has issued a "Memorandum on the Teaching of Geometry in Secondary Schools" which covers essentially the same ground as the circular on geometry issued five years ago by the Board and now out of print.

A large volume entitled "List of Prime Numbers from 1 to 10,006,721" by Professor D. N. Lehmer, University of California, was recently published by the Carnegie Institution of Washington. The last four digits of the primes are given in the body of the table and the remaining digits are at the top and bottom of the columns. Five thousand primes are listed on each page, and the table gives the rank of any prime in the series of primes. The same precautions were taken in printing these tables as were observed in printing the factor tables for the first ten million numbers, which were prepared in 1909 by the same author under the auspices of the Carnegie Institution. By the publication of these tables Professor Lehmer has rendered an important service to mathematics, in view of their great accuracy and convenient arrangement.

The last volume (XXI) of the *Proceedings* of the Society for the Promotion of Engineering Education contains, among numerous articles of a more special nature, three which have a general interest and which perhaps deserve mention here. The titles are

(1) How Can the Colleges and the Industries Cooperate?
By Ivy L. Lee, Executive Assistant, Penn. R. R. Co.

(2) [Same title as (1)].
By Edward D. Sabine, Terminal Engineer, N. Y. C. & H. R. R.

(3) Recommendations Concerning the Units of Force.
By Professor E. V. Huntington, Harvard University.

The first two will be interesting and instructive to upper classmen in schools of engineering. The third is especially recommended to teachers of physics in secondary schools.

The universities of Great Britain, according to announcements, opened as usual this academic year with probably half the regular attendance, one half being in active military service. The German universities also are reported to have opened.

Professor F. R. Moulton published in *Popular Astronomy* for August–September a very lucid and comprehensive analysis of the works of the noted American astronomer, Dr. George W. Hill, who died April 16, 1914.

"Problemi della Scienza," published in 1906 by F. Enriques, professor of projective and descriptive geometry in the University of Bologna, has been translated into Englisb by Katharine Royce; the book is published by the Open Court Publishing Company under the title "Problems of Science."

A press dispatch appeared in August describing the unusual mental calculations carried out before a class in psychology at the summer session of the University of California by Arthur A. Gamble, "a 21-year-old student of the University of Chicago." A letter of inquiry to Professor Warner Brown of the department of psychology in the University of California brought an authoritative account from which the following interesting extracts are taken.

"He is not a college student.[1] . . . He knows no mathematics beyond simple arithmetic—does not even know what logarithms are or how they are used. He does know several short-cuts in multiplication and division (multiplying from the left, etc.). His *forte* is factoring. He can give offhand (5 seconds or less) two numbers which when multiplied together make any 5- or 6-place number which is proposed; he often gives two numbers which nearly make the given number, with a statement of the remainder. As you see, this depends upon the knowledge of a rather extended multiplication table, say up to 500. He is familiar enough with the products of numbers to be able to give squares and cubes quickly. . . .

"Division he accomplishes by tentative multiplication, which is easier for him. Cube roots are not easy for him; he makes a large number of guesses and tries them out by multiplication. . . . He knows the powers of 2 ("doubling a penny") up to the 32d and can reel them off as fast as he can speak.[2]

"He knows the data necessary for calendar computations and gives the day of the week for any date very quickly.

"He *cannot* add very rapidly.[3] His memory for *digits* is not unusual, but he can hold in memory a large collection of *numbers*. . . . He has an unbounded

[1] He visited the University of Chicago and members of the mathematical staff tried him out with the same conclusions as reached by Professor Brown. They advised him to return to his home in Rochester, N. Y., and continue his regular high school course rather than continue to develop his freakish propensities. EDITORS.

[2] Evidently memorized from the famous horse shoe problem. ED.

[3] The newspaper item stated that Dr. Brown handed simultaneously to Mr. Gamble and an expert adding machine operator a list of numbers to be added, all of four or more digits, Mr. Gamble obtaining the correct sum more quickly than the mechanical adder. ED.

interest in numbers as numbers. The result is that he has come to individualize thousands of them to almost the same extent that you or I individualize the numbers below 100; he knows their relationships in the arithmetical family."

L'Enseignement Mathematique is doubtless the leading journal devoted to the pedagogy of mathematics. This appears under the joint editorship of C. A. Laisant of Paris and H. Fehr of Geneva, with the collaboration of A. Buhl of Toulouse; it is published by Gauthier-Villars of Paris and Georg and Cie of Geneva. The publication is now in the sixteenth year; the subscription price is 15 francs per year (Stechert and Co., West 25th St., New York City). Each volume consists of about 500 pages.

Probably the general nature of the journal is best illustrated by a summary of the articles which appear in the current volume (1914). The first number, issued January 15, 1914, of some 80 pages, includes articles on non-Euclidean geometry as applied to the theory of relativity, on multiple equalities, on the integration of the equations for the movements of a planet about the sun, on some points on the theory of sets, on transcendental plane curves given by equations in which the variables are separable, on a double system of lines on a surface, on an application of the rule of false position, and on new formulas for Heronian triangles. These articles by French, German, Italian, and Russian authors are all written in French. In addition to the articles, space is devoted to current events, to notes and documents, to reviews, and bibliography. The second number is of the same general nature as the first, while the third, fourth, and fifth issues are devoted almost entirely to a report of the International Conference on the teaching of mathematics which was held at Paris, April 1–4, 1914.

The titles of the articles, above-mentioned, show clearly that the appeal is made not to the beginner in mathematics but to the real student of the science. Historical articles are of rather infrequent occurrence. The notes and documents in recent issues have been devoted in large measure to the activities of the International Commission, while the treatment of reviews and bibliography of current literature in the field does not vary greatly from the procedure in American journals.

ERRATA.

Page 164, line 4 up. For J. A. Colson, *read* J. W. Clawson. Page 190, under Algebra. For **416** *read* **416A**, and for **417** *read* **417A**. Page 258, line 8 up. For 2^{2a_3} *read* 2^{2a_3} Page 259, line 2 down. For $\chi^{p(p-1)}$ *read* $x^{p(p-1)}$. Page 290, line 2 up. For dr^{n-1} *read* $d^{n-1}r$.

Also on page 259, the author wishes to substitute for the paragraph (4) the following: "To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was a discovery claimed by Pappus (about 300 A. D.).[1] Nicomedes (about 180 B. C.) used the curve for the duplication of the cube[2] and we have only the assertion of Proclus that Nicomedes also used it for the trisection of an angle."

Page 299, last line, last term. For $(-1)^k$ *read* $(-1)^{k-1}$.

INDEX TO VOLUME XXI

AUTHORS

ARCHIBALD, R. C. Remarks on Klein's "Famous Problems of Elementary Geometry," 247.

BARTON, S. G. A Simple Method of Constructing the Normals to a Parabola, 180.

——— Some Properties of the Normals to a Parabola, 182.

BAUER, GEO. N. See REVIEWS under Moritz.

BEKE, E. The Paris Report on Calculus in Secondary Schools, 324.

BRENKE, W. C. See REVIEWS under Burkhardt.

BRUCE, R. E. A Theorem in the Modern Geometry of the Abridged Notation, 77.

BUSSEY, W. H. The Tactical Problem of Steiner, 3. See REVIEWS under Dickson.

CAIRNS, W. DEW. See REVIEWS under Franklin, Michaelis, Proceedings.

CAJORI, F. The Napier Tercentenary Celebration, 321.

CARMICHAEL, R. D. See REVIEWS under Whitford.

CLARKE, E. H. A Formula for the sum of a Certain Type of Infinite Power Series, 292.

COOLIDGE, J. L. A Simple Algebraic Paradox, 184.

COREY, S. A. A Method of Solving Numerical Equations, 290.

DICKSON, L. E. On the Trisection of an Angle and the Construction of Regular Polygons of 7 and 9 sides, 259.

ELDER, F. S. Note on a Memory Device for Hyperbolic Functions, 51.

EMCH, A. On some Geometric Properties of Circular Transformations, 139.

FIELD, P. See Ziwet.

HARDING, A. M. The Apparent Size of a Cube, 209.

HARTWELL, G. W. See REVIEWS under Collins.

HUNTINGTON, E. V. The Theorem of Rotation in Elementary Mechanics, 315.

JOFFE, S. A. Concerning the Regular Inscribed Hexagon, 147.

KARPINSKI, L. C. The Algebra of Abu Kamil, 37.

——— See REVIEWS under Smith, Mikami, Klein.

KEMPNER, A. J. A Curious Convergent Series, 48.

KENYON, A. M. On the use of Partial Derivatives in Plotting Curves from their Equations, 288.

LEHMER, D. N. A Theorem in Number Theory connected with the Binomial Formula, 15.

——— On a Purely Projective Basis for the Theory of Involution, 185.

——— On a Special Case of the Tetrahedral Complex, 287.

——— See REVIEWS under Wilczynski.

LORIA, G. Sur un Paradoxe Algébrique Apparent, 327.

MCNATT, J. Q. A Geometrical Discussion of the Regular Inscribed Heptagon, 13.

MILLER, G. A. Groups of Figures of Elementary Geometry, 285.

MORGAN, F. M. A Plea for less Formal Work in Mathematics, 116.

NEWKIRK, B. L. See REVIEWS under Capito.

OSBORNE, G. A. A Problem in Number Theory, 148.

PONZER, E. W. An Equation Balance for Classroom Use, 283.

PUTNAM, T. M. Residues of Certain Sums of Powers of Integers, 220.

REMICK, B. L. Note on Perfect Magic Squares for 1914, 151.

ROEVER, W. H. Optical Interpretations in Higher Geodesy, 69.

SHUMWAY, R. R. See REVIEWS under Long.

SLAUGHT, H. E. Retrospect and Prospect, 1.

SLOBIN, H. L. Note on Certain Algebraic Equations, 113.

SNYDER, V. See REVIEWS under Ziwet.

SPUNAR, V. M. The Perfect Magic Squares for 1914, 50

STROMQUIST, C. E. See REVIEWS under Hatton.

SWIFT, E. See REVIEWS under Dickson.

TRIPP, M. O. An Application of Partial Derivatives to the Ellipse, 15.

UNDERHILL, A. L. See REVIEWS under Longley.

WELLS, AGNES E. See REVIEWS under Ford.

WOODS, B. M. The Construction of Conics under given Conditions, 173.

ZIWET, A., and FIELD, PETER. Note on Plane Kinematics, 105.

ZELENY, A. See REVIEWS under Galilei.

PROBLEMS AND MISCELLANEOUS QUESTIONS

ALTSHILLER, N. Problem proposed in Geometry, 331.

BAKER, R. P. Problems proposed and solved in Geometry and Calculus, 22, 231, 273, 274, 301.

BARNETT, I. A. Problems proposed and solved in Algebra, Calculus and Mechanics, 92, 162, 163, 191, 227, 235.

BARTON, S. G. Problems proposed and solved in Geometry and Mechanics, 62, 191.

BELL, E. T. Problems proposed in Number Theory, 23, 55, 91, 123, 130, 131, 132, 157, 227, 238.
BEMAN, W. W. Problems proposed and solved in Algebra and Calculus, 23, 235.
BIGBEE, J. A. Problems proposed in Geometry, 22, 160.
BIGELOW, H. T. Problem solved in Geometry, 304.
BLAKESLEE, T. M. Problems solved in Algebra, Geometry, and Calculus, 192, 194, 340.
BRENKE, W. C. Problem solved in Algebra, 334.
BUSSEY, W. H. Problems proposed in Algebra and Geometry, 21, 54, 125, 158.
CAIRNS, W. D. Problems proposed in Algebra, Calculus, and Number Theory, 90, 122, 299, 332.
CAJORI, F. Problem proposed in Geometry, 21.
CAMPBELL, J. W. Problem solved in Algebra, 125.
CAPRON, P. Problems proposed and solved in Algebra, Geometry and Calculus, 227, 267, 301, 305, 336.
CARIS, A. G. Problems solved in Algebra and Calculus, 159, 192, 336.
CARMICHAEL, R. D. Discussion on Synthetic Projective Geometry, 63.
——— Problems proposed and solved in Algebra, Geometry, Calculus, and Number Theory, 21, 22, 23, 24, 54, 55, 61, 91, 123, 127, 155, 157, 158, 159, 165, 191, 192, 199, 200, 234, 238, 334.
CLARKE, E. H. Problems proposed in Algebra, 266, 299.
CLAWSON, J. W. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 125, 126, 127, 163, 164, 228, 231, 233, 235, 236, 237, 274, 275, 301, 307, 340.
COFFIN, L. M. Problems solved in Algebra, 58, 91.
COHEN, L. See Miscellaneous Questions, 29.
COLSON, J. A. Problem solved in Algebra, 25.
COREY, S. A. Problems proposed in Algebra and Geometry, 89, 190.
CULLUM, W. Problems solved in Algebra, 158, 191.
DICKSON, L. E. Problem proposed in Number Theory, 97.
DRANE, W. H. Problem proposed in Mechanics, 55.
DRAXTEN, N. A. Problem solved in Number Theory, 99.
EELLS, W. C. Problems solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 25, 26, 58, 91, 97, 127, 158, 163, 228, 231, 233, 235, 269, 301, 307, 336.
EGERER, H. Problem solved in Calculus, 59.
ESCOTT, E. B. Problems proposed and solved in Algebra, Calculus, Mechanics, and Number Theory, 24, 25, 62, 89, 91, 98, 99, 127, 230, 231.
FEEMSTER, H. C. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 25, 26, 28, 29, 58, 91, 92, 97, 126, 132, 159, 163, 268, 301, 305, 307, 334, 336.
FINKEL, B. F. Problems proposed and solved in Geometry, Calculus, and Mechanics, 90, 122, 156, 157, 161, 191, 195, 236, 267.
FLANAGAN, C. E. Problems proposed and solved in Algebra, and Geometry, 156, 273, 307.
GIBSON, EMMA M. Problems proposed and solved in Algebra, Geometry, Calculus, and Mechanics, 121, 122, 127, 156, 157, 160, 228, 268, 335.
GITHENS, C. E. Problems proposed and solved in Algebra, Geometry, and Number Theory, 91, 97, 99, 121, 166, 201, 268, 301, 307, 334, 336.
GONZALES, J. G. Problem solved in Calculus, 236.
GRABER, M. E. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 62, 94, 126, 127, 235.
GREENSTREET, W. J. Problems proposed in Mechanics, 96, 340.
HANSON, H. O. Problem proposed in Algebra, 190.
HARDING, A. M. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 23, 25, 27, 28, 62, 91, 95, 96, 125, 127, 129, 158, 159, 160, 163, 164, 191, 192, 193, 195, 197, 198, 228, 233, 234, 235, 237, 268, 305, 307, 336, 340. See Miscellaneous Questions, 342
HARTWELL, G. W. Problems solved in Algebra, Geometry, and Calculus, 126, 127, 158, 159, 161, 235, 236, 268, 301, 307, 336.
HASKINS, C. N. See Miscellaneous Questions, 30.
HEAL, W. E. Problem proposed in Number Theory, 62.
——— See Miscellaneous Questions, 278.
HOLMES, A. H. Problems proposed and solved in Algebra, Geometry, Calculus, and Number Theory, 26, 28, 57, 62, 91, 92, 97, 99, 128, 192, 267.
HOPKINS, G. I. Problem solved in Algebra, 273.
HORNE, C. E. Problems proposed and solved in Algebra, and Mechanics, 90, 125, 163, 307.
HORNUNG, C. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 61, 126, 127, 159, 160, 228, 235.
JOFFE, S. A. Problems proposed and solved in Algebra, 230, 331.
KELLEY, D. F. Problems solved in Algebra and Geometry, 28, 58, 125.
KEMPNER, A. J. Problems proposed in Geometry and Number Theory, 55, 90, 123, 304, 307.
KRAMER, B. Problem solved in Algebra, 301.
KRETH, D. Problems proposed and solved in Algebra and Geometry, 22, 126, 233.
LAWRENCE, J. F. Problem proposed in Algebra, 157.

LEBOLD, W. R. Problem proposed in Calculus, 194.
LEFSCHETZ, S. Problem proposed in Geometry, 25.
LEHMER, D. N. Discussions on Synthetic Projective Geometry, 30; Culture Courses, 133.
LEVY, H. A. Problem solved in Algebra, 25.
LIBBY, B. L. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 60, 127, 158, 159, 161, 163, 199, 234, 235.
LIGHT, G. H. Problems proposed and solved in Mechanics, 61, 62.
LOCHNER, R. P. Problems proposed in Calculus, Mechanics, and Number Theory, 22, 55, 58, 161, 163, 165, 237, 275.
LONG, EDITH. Coordinated Courses in High School and College Mathematics, 239.
MACINNES, C. R. See Miscellaneous Questions, 278.
MACKAY, D. L. Problem solved in Geometry, 95.
MAGLOTT, EVA S. Problem proposed in Calculus 95.
MARTIN, A. Problems solved in Algebra and Number Theory, 98, 126, 165, 199.
MASON, T. E. Problems proposed and solved in Algebra and Number Theory, 28, 92, 99, 130, 227, 267.
MATHEWS, R. M. Problems proposed and solved in Algebra and Geometry, 25, 90, 91, 233.
MATHEWSON, L. C. Problem solved in Number Theory, 29.
MATZ, F. P. Problem proposed in Mechanics, 198.
MCCARTY, A. L. Problems solved in Algebra, Geometry, and Calculus, 23, 125, 159, 235, 306, 340.
MCCUE, M. J. Problem solved in Mechanics, 62.
MILLER, G. A. Course in History of Mathematics, 202.
MILLS, C. N. Problems proposed and solved in Algebra, Geometry, Calculus, and Mechanics, 90, 121, 158, 159, 160, 190, 226, 228, 232, 235, 237, 300, 301, 305, 307, 331, 332, 336.
MIRICK, G. R. Problem solved in Algebra, 301.
MOORE, F. C. Problem solved in Calculus, 163.
MORGAN, F. M. Problems proposed and solved in Algebra, Geometry and Calculus, 25, 27, 54, 57, 58, 91, 159, 160, 161, 192, 193, 228, 236.
MORRIS, R. Problems solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 91, 127, 129, 163, 201, 235, 236, 237.
MOULTON, E. J. Problems proposed and solved in Algebra, 89, 228, 229.
MOULTON, F. R. See Miscellaneous Questions, 133.
NAUER, A. R. Problems solved in Algebra and Calculus, 91, 126, 158, 235, 301.
OLSON, H. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 25, 26, 91, 99, 123, 126, 127, 159, 160, 192, 199, 226, 228, 233, 234, 268, 300, 305, 307, 336.
OSBORN, G. A. Problem proposed in Algebra, 226.
O'SHAUGHNESSY, L. Problems solved in Algebra and Number Theory, 191, 307.
PEÑALVER, P. Problems solved in Algebra, Calculus, and Mechanics, 96, 126, 163, 234, 235.
PRIME, H. Problem proposed in Algebra, 56.
REAVES, S. W. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 25, 28, 58, 91, 92, 97, 126, 127 129, 160, 163, 192, 195, 198, 227, 233, 235, 236, 237, 305.
REISLER, F. C. Problems solved in Algebra, Calculus, and Mechanics, 92, 162, 163, 235.
REYNOLDS, K. Problem solved in Geometry, 127.
RICHARDSON, A. J. Problem proposed in Algebra, 156.
RILEY, J. L. Problems solved in Algebra, Geometry, and Calculus, 91, 127, 233, 301, 307.
RODGERS, T. G. Discussion on Synthetic Projective Geometry, 32.
ROESER, H. Problem solved in Geometry, 95.
ROUILLION, L. Problem proposed in Geometry, 331.
RUST, F. Problem solved in Calculus, 235.
SANDERS, J.E. Problem solved in Mechanics, 62.
SCHEFFER J. Problems proposed and solved in Geometry and Mechanics 27, 62, 96.
SCHMALL, C. N. Problems proposed and solved in Algebra, Geometry, Calculus, and Mechanics, 21, 22, 25, 26, 54, 55, 58, 90, 95, 121, 122, 123, 125, 127, 129, 155, 156, 158, 162, 163, 164, 191, 197, 198, 227, 228, 233, 235, 266, 267, 300, 301, 305, 306, 331, 334, 336, 340.
SCHMIEDEL, O. Problem solved in Algebra, 159.
SCHUYLER, E. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 23, 25, 26, 27, 28, 54, 58, 60, 61, 91, 95, 96, 122, 123, 125, 126, 127, 132, 158, 159, 163, 190, 198, 235, 301, 307, 336.
SHIVELY, L. S. Problem solved in Algebra, 25.
SIMONS, L. G. Discussion on Synthetic Projective Geometry, 100.
SLOBIN, H. L. Problems proposed and solved in Algebra and Calculus, 121, 124, 235, 332.
——— See Miscellaneous Questions, 201.
SMITH, J. B. Problems proposed and solved in Algebra, Geometry, Calculus, and Mechanics, 121, 127, 130, 156, 159, 163, 199, 235, 237, 305.
SOSNOW, G. Y. Problems solved in Algebra, 25, 158.
SPUNAR, V. M. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 54, 91, 94, 97, 121, 126, 129, 164, 228, 300, 331.

STECK, C. C. Problems solved in Calculus, 235, 236.
STRATTON, W. T. Discussions on Vocational Training, 166; Course in History of Mathematics, 202.
STROMQUIST, C. E. Discussion on Synthetic Projective Geometry, 31.
SWIFT, E. Problems proposed and solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 157, 158, 159, 191, 192, 197, 198, 228, 233, 238, 307.
TREFETHEN, H. E. Problems proposed and solved in Algebra, Geometry, Mechanics, and Number Theory, 57, 58, 91, 94, 99, 123, 129, 156, 266.
TREFETHEN, MRS. H. E. Problem solved in Mechanics, 62.
UHLER, H. S. See Miscellaneous Questions, 278, 342.
WALSH, P. Problem proposed in Number Theory, 332.
WATSON, W. L. Problems proposed in Geometry, 122, 267, 335.
WHITFORD, E. E. Problems solved in Algebra, Geometry, and Number Theory, 200, 232, 273.
WILSON, A. H. Remark, 340. Problem solved in Mechanics, 341.
WILSON, E. B. Problem solved in Algebra, 335.
YANNEY, B. F. Problems solved in Number Theory, 29, 62, 97, 99, 131.
ZERR, G. B. M. Problem proposed in Mechanics, 340.

BOOK REVIEWS

Ammerman, C. See Ford.
Brenke, W. C. See Long, Edith.
Burkhardt, H. Theory of Functions of a Complex Variable. Translated by S. E. Rasor. W. C. BRENKE, 118.
Capito, C. A. A. A Textbook of Mathematics and Mechanics. B. L. NEWKIRK, 223.
Charles, R. L. See Franklin.
Collins, J. V. Advanced Algebra. GEO. W. HARTWELL, 117.
Dickson, L. E. Elementary Theory of Equations. ELIJAH SWIFT, 264.
———, and Osgood, W. F. The Madison Colloquium Lectures on Mathematics. W. H. BUSSEY, 263.
Ford, W. B., and Ammerman, C. Plane and Solid Geometry. AGNES E. WELLS, 222.
Franklin, W. S., MacNutt, B. and Charles, R. L. An Elementary Treatise on Calculus. W. DE W. CAIRNS, 18.
Galilei, Galileo. Dialogues concerning two Sciences. A. ZELENY, 328.
Hatton, J. L. S. The Principles of Projective Geometry. C. E. STROMQUIST, 119.
Kliem, Fritz. Archimedes' Werke. L. C. KARPINSKI, 263.
Long, Edith, and Brenke, W. C. Algebra, First Course. R. R. SHUMWAY, 17.
Longley, W. R. Tables and Formulas for Solving Numerical Problems in Analytic Geometry, Calculus and Applied Mathematics. A. L. UNDERHILL, 52.
MacNutt, B. See Franklin.
Michaelis, L. Mathematik für Biologen und Chemiker. W. De W. CAIRNS, 52.
Mikami, Yoshio, The Development of Mathematics in China and Japan. L. C. KARPINSKI, 153.
——— See Smith, D. E.
Moritz, R. E. Memorabilia Mathematica. Geo. N. BAUER, 296.
Osgood, W. F. See Dickson.
Proceedings of the Society for the Promotion of Engineering Education. W. DE W. CAIRNS, 53.
Rasor, S. E. See Burkhardt, H.
SLAUGHT, H. E. See Wilczynski.
Smith, D. E. The Teaching of Arithmetic. L. C. KARPINSKI, 85.
———, and Mikami, Yoshio. A History of Japanese Mathematics. L. C. KARPINSKI, 151.
Wilczynski, E. J. Plane Trigonometry and Applications. Edited by H. E. Slaught. D. N. LEHMER, 329.
Whitford, E. E. The Pell Equation. R. D. CARMICHAEL, 188.
Ziwet, A., and Hopkins, L. A. Analytic Geometry and Principles of Algebra, VIRGIL SNYDER, 86.

NEW BOOKS

Bernstein, B. A. A Complete Set of Postulates for the Logic of Classes expressed in terms of the Operation "Exception," 279.
Blichfeldt, H. F. The Theory of Collineation Groups, 345.
Burkhardt, H. Theory of Functions of a Complex Variable, Translated by S. E. Rasor, 32.
Cajori, F. History of Elementary Mathematics, 204.
Carmichael, R. D. The Theory of Numbers, 204.
Coates, J. V. H. A First Book of Geometry, 244.
Cordeiro, F. J. B. The Gyroscope; its Theory and Practice, 32.
Crew, H., and De Salvio, A. Dialogues Concerning two New Sciences, 204.
Davis-Brenke, Answers to Calculus, 103.
Dickson, L. E. Elementary Theory of Equations, 205.
Dingeldey, F. Sammlung von Aufgaben, 33.
Dowling, L. W., and Turneaure, F. E. Plane Analytic Geometry, 204.
Encyclopedie des Sciences Mathematiques, 33, 168, 205, 245.
Encyklopädie der Mathematischen Wissenchaften, 205, 243, 280.

Enriques, F. Problemi della Scienza. Translated by Katharine Royce, 346.
Euler, Complete Works, 66, 205.
Favaro, A. Per la biografia di Niccolò Tartaglia, 135.
Godfrey and Siddons. A Shorter Geometry, 245.
Kennelly, A. E. The Hyperbolic Functions of Complex Variables, 169.
Klein, Felix. Lectures on the Icosahedron. Translated by G. G. Morrice, 344.
Lehmer, D. N. List of Prime Numbers, 345.
Manchester, R. E. The Teaching of Mathematics, 344.
Manning, H. P. The Geometry of Four Dimensions, 135.
Mikami, Y. See Smith, D. E.
Milliken, R. A. Isolation and Measurement of the Electron, 279.
Mitzscherling, A. Das Problem der Kreisteilung, 67.
Moritz, R. E. Memorabilia Mathematica or the Philomath's Quotation Book, 135, 169.
Moulton, F. R. Introduction to Celestial Mechanics, 204.
Raymond. Plane Surveying for use in the Class-room and Field, 279.
Russel, B. See Whitehead.
Silberstein, L. The Theory of Relativity, 344.
Sisam, C. H. See Snyder.
Slaught, H. E. See Wilczynski.
Smith, D. E., and Mikami, Y. History of Japanese Mathematics, 135.
Snyder, V., and Sisam, C. H. Analytic Geometry of Space, 204.
Stamper, A. W. Teaching of Arithmetic, 205.
Thorndike, E. L. Introduction to the Theory of Mental and Social Measurements, 310.
Turneaure, F. E. See Dowling.
Webb, W. L. Biography of T. J. J. See, 66.
Weber, H. Kleines Lehrbuch der Algebra, 167.
Whitehead, A. N., and Russel, B. Principia Mathematica, 103.
Wilczynski, E. J., and Slaught, H. E. Trigonometry, 206.

PERSONAL MENTION

Albeggiani, M., 170; Albert, O. W., 242; Allen, E. S., 279, 282; Anderegg, F., 311; Andoyer, H., 137, 169, 322; Appell, P., 68; Archibald, R. C., 242, 310, 314; Austin, C. M., 138.

Babb, M. J., 103; Bagnera, G., 170; Baker, H. F., 136; Ball, R. S., 66, 136; Bardeen, C. W., 344; Barker, J. F., 137; Bartoo, G. C., 207; Barus, C., 314; Bauer, G. N., 204, 312; Baur, L., 244; Bauschinger, J., 137, 322; Beetle, R. D., 282; Beke, E., 68, 324; Bell, G., 311; Beman, W. W., 168, 206; Benedict, Susan R., 244, 282; Bernstein, B. A., 279, 282; Bill, E. G., 168; Birkhoff, G. D., 312; Blichfeldt, H. F., 345; Bliss, G. A., 103, 171, 282; Bloch, E., 170; Bloche, C., 68; Blumberg, H., 33; Boehm, K., 66; Bolza, O., 33; Borel, E., 68; Bortolotti, E., 103; Bourlet, C., 33, 68; Bouton, C. L., 203; Bradshaw, J. W., 168; Brenke, W. C., 33, 103; Breslich, E. R., 138; Brookman, Miss T., 243; Brown, G. L., 136; Brown, H., 137; Brown, J. C., 103; Brown, W., 346; Brush, C. F., 311; Bryan, G. H., 279; Buhl, A., 347; Bullard, J. A., 282; Burgess, R. W., 282; Burkhardt, H., 32; Bussey, W. H., 244; Byerly, W. E., 310.

Cairns, W. D., 203, 279; Cajori, F., 34, 135, 137, 168, 203, 204, 281, 311, 314, 322, 347, 348; Carmichael, R. D., 32, 204; Chamberlin, T. C., 66; Coates, J. V. H., 244; Cobb, C. W., 102; Coffman, L. D., 34; Comstock, C. E., 34; Conway, A. W., 35; Coradi, G., 323; Cordeiro, F. J. B., 32; Crew, H., 204; Crommelin, A. C. D., 66; Cunningham, A., 281; Cunningham, E., 208; Curtiss, D. R., 171, 282.

Davis, E. W., 103; Davis, N. F., 313; Dean, G. R., 343; de Franchis, M., 170; De Salvio, A., 204; Devereaux, Miss M., 137; Dexter, 67; Dickson, L. E., 171, 205, 206, 263, 282; Dines, L. L., 343; Dingeldey, F., 33; D'Ocagne, M., 68, 208, 322; Dodd, E. L., 33, 66, 203; Dowling, L. W., 204; Duhem, P., 244; Duval, E. P. R., 343.

Edington, W. E., 311; Eells, W. C., 102; Eiesland, J., 312; Eikenberry, W. L., 344; Eliot, C. W., 243; Emch, A., 135; Eneström, G., 202; Enriques, F., 346; Escott, E. B., 242, 281; Eshleman, J. D., 206, 345; Evans, G. W., 206.

Faber, G., 66; Fauquembergue, E., 281; Faunce, President, 313, 314; Favaro, A., 135, 170; Fehr, H., 347; Ferguson, J. C., 322; Ferry, F. C., 206, 243, 311; Finkel, B. F., 208; Fiske, T. S., 169; Ford, A. H., 135; Ford, W. B., 168, 207, 242; Forsyth, 168; Franklin, W. S., 53; Fréchet, M., 65, 168, 345; Funk, C. F. B., 67.

Gaba, M. G., 243, 280; Gain, J. W., 282; Galajikian, H., 282; Galbraith, A. V., 206; Gamble, A. A., 346; Garrett, W. H., 280; Gay, H. S., 322; Gebbia, M., 170; Gee, E. T., 207; Gibson, G. A., 137, 322; Gifford, Mrs., 322; Gilmore, Miss S., 36; Githens, C. E., 205; Glaisher, J. W. L., 66, 137, 321, 323; Godfrey, C., 245, 326; Gossard, H. C., 282; Graham, Effie, 103; Gray, Ella D., 311; Green, G. N., 203; Grimsehl, E., 171; Gronwall, T. H., 206, 242, 344; Guccia, G. B., 170; Guthe, K. E., 244.

Hadamard, J., 68; Halsted, G. B., 153; Hart, J. N., 311; Hart, W. L., 282; Haskell, M. W., 173; Hawkes, H. E., 169; Hedrick, E. R., 33, 222, 245; Hemke, P. E., 245; Heuter, R. R., 53; Hildebrandt, T. H., 168; Hill, G. W., 346; Hill, L., 203; Hime, H. W. L., 244; Hine, Miss K. G., 207; Hirsch, S. A., 244; Hobson, E. W., 321; Holman, W. F., 244; Holroyd, Miss I. E., 242; Horsburgh, E. M., 323; Hudson, 322; Huntington, E. V., 345; Hutchinson, A., 322; Hyde, E. W., 67.

Ingold, L., 33.

Jackson, C. S., 171; Jackson, D., 312;

Johnson, R. A., 310; Jordan, W. L., 34; Jourdain, P. E. B., 170; Judd, H., 53.

Karpinski, L. C., 168, 206, 207; Kellogg, O. D., 33; Kells, L. M., 282; Kempner, A. J., 33; Kennelly, A. E., 169; Keyser, C. J., 169, 170, 242; Kingston, H. R., 280; Kircher, E. A. T., 204, 282; Kirk, H. C., 102; Klein, F., 68; Knott, C. G., 35; Krathwohl, W. C., 282; Küschke, C., 203; Kusterman, W. W., 242.

Laisant, C. A., 68, 347; Lamb, Henry, 66; Lamb, Horace, 66; Landau, E. G. H., 280; Lang, 137; Laplace, P. S., 168; Laves, K., 171; Lebon, E., 313; Lecat, M., 33; Lee, I. L., 345; Lefschetz, S., 33; Lehmer, D. N., 36, 243, 345; Lennes, N. J., 203; Lietzmann, W., 171; Little, A. G., 244; Lloyd, 244; Loria, G., 170; Lovett, E. O., 104; Lovitt, W. V., 280, 282; Lunn, A. C., 34, 171.

Macdonald, 137; MacGillivray, A. O., 169; MacNutt, B., 53; Magruder, W. T., 53; Manchester, R. E., 344; Manning, H. P., 135; Markley, J. L., 168; Marshall, W., 242, 310; Martin, A., 66; Mason, T. E., 242, 282; Mathewson, L. C., 282; Mauckley, S. J., 137; McMurray, F. M., 222; McNatt, J. Q., 147; Meissner, W., 281; Merriman, M., 204; Meserve, C. D., 206; Metzler, W. H., 312; Mikami, Yoshio, 67, 135; Miller, A. L., 103, 168; Miller, Bessie I., 282; Miller, G. A., 36, 103, 205, 280; Miller, J. B., 67; Millikan, R. A., 279; Mills, C. N., 136; Milne, J. R., 322; Milne, W. J., 310; Miser, W. L., 282; Mitchell, H. B., 169; Mitzscherling, A., 67; Moore, C. N., 170, 171, 282, 344; Moore, E. H., 172, 205; Morehouse, D. W., 282; Morgan, F. M., 168; Moritz, R. E., 135, 169; Morrice, G. G., 344; Morris, C. C., 137; Morrison, F. M., 282; Morse, F. P., 206; Moulton, E. J., 282; Moulton, F. R., 33, 172, 204, 246, 346; Moulton, J. F., 137; Moulton, Lord, 321; Mourad, Salik, 322; Muir, M. M. P., 244; Myers, G. W., 135, 172.

Nicholson, T. W., 67; Norris, H. H., 53.

Osgood, W. F., 263, 310; Ott, A., 323.

Pascal, E., 137; Pearson, K., 137, 345; Perron, O., 135; Picard, E., 104; Picavet, F., 244; Peirce, B. O., 102, 310; Pierce, Miss H. R., 206; Pitcher, A. D., 168; Pitrè, G., 170; Plimpton, G. A., 206; Poincaré, H., 312, 313; Poincaré, L., 68; Pollock, Jane V., 34; Poor, V. C., 168; Putnam, T. M., 204.

Raffaele, F., 170; Ranum, A., 204; Rasor, S. E., 32, 137; Raymond, W. G., 279; Reinert, W. A., 242; Remsen, I., 169; Richardson, R. G. D., 314; Richmond, H. W., 208; Riley, J. L., 343; Robbins, C. K., 242; Robbins, R. B., 282; Rodgers, T. G., 1; Roever, W. H., 172; Rowe, J. E., 313; Royce, Katharine, 346; Ruffini, P., 103; Runge, C., 204; Running, T. R., 207; Russell, B., 32, 103.

Sabine, E. D., 345; Sabine, W. C., 310; Sandys, J. E., 244; Sarton, G., 170; Schear, E. W. E., 137; Schlesinger, F., 33; Schooling, 322; Schotten, H., 171; Schwatt, I. J., 206; Schweitzer, A. R., 33; See, T. J. J., 66; Shenton, W. F., 279, 282; Sheppard, W. F., 322; Short, R. L., 34; Shugert, S. P., 282; Siddons, A. W., 245; Silberstein, L., 244; Simon, M., 280; Sinclair, Miss M. E., 206; Sisam, C. H., 204; Slaught, H. E., 137, 138, 172, 206, 282; Slobin, H. L., 201, 244, 312; Smith, D. E., 135, 137, 169, 242, 244, 322, 326; Smith, E. R., 313; Smith, G. A., 34; Smith, S. E., 206; Snyder, V., 204; Sommers, J., 280; Sommerville, D. M. Y., 35, 322, 323; Spunar, V. M., 151; Staeckel, P., 68; Stager, H. W., 36, 244; Stamper, A. W., 205; Stecker, H. F., 313; Steggall, 137; Stetson, J. M., 282; Stevens, R., 104; Stone, R. B., 242; Stouffer, E. B., 345; Sunagawa, Seiich, 204; Sundmann, M., 104; Swartz, W. G., 34; Swift, E., 244.

Tappan, Anna H., 282; Tatlock, 244; Teubner, B. G., 171; Thorndike, E. L., 170, 245, 310; Tonni-Bazza, V., 170; Tripp, M. O., 280; Trowbridge, J., 310; Turneaure, F. E., 204; Tyler, H. W., 206.

Vacca, G., 321; VanVleck, E. B., 33, 104, 311, 326; Venturi, A., 170; Vogl, S., 244; Volterra, V., 168.

Warren, L. A. H., 282; Watson, G. N., 281; Webb, H. E., 32; Webb, W. L., 66; Weber, H., 167; Weeks, Miss E. A., 33; Wells, Miss M. E., 206; Wells, V. H., 244; Westfall, W. D., 33; Wheelock, C. F., 103; White, H. S., 104; Whitehead, A. N., 103, 281; Whitford, E. E., 343; Whitney, A. W., 33; Whittaker, E. P., 35; Whittaker, E. T., 137, 208; Wiedemann, E., 244; Wilczynski, E. J., 206; Wiley, F. B., 243, 280; Williams, K. P., 279; Williams, L. E., 206, 245; Withington, E., 244; Woodward, C. M., 136; Woodward, R. S., 204; Wright, H. C., 138; Wurmschmidt, J., 244.

Yocum, A. D., 104, 311; Young, J. W. A., 36, 172, 282; Young, Mabel M., 282.

Zacharias, M., 280; Zeis, H. C., 34; Ziwet, A., 206.

JOURNALS, ASSOCIATIONS, etc.

American Association for the Advancement of Science, 33, 104, 205. American Mathematical Society, 205, 313. Annals of Mathematics, 246. Association of Mathematics Teachers of New Jersey, 313. Association of Ohio Teachers of Mathematics and Sciences, 137, 172. Association of Teachers of Mathematics in the Middle States and Maryland, 103, 169.

Bulletin des Sciences Mathématiques, 104. Bulletin of the American Mathematical Society, 310.

California High School Teachers' Association, 36, 243. Central Association of Science and Mathematics Teachers, 34, 135, 243, 344. Chicago Section of the American Mathematical Society, 34, 135, 310. Circolo Matematico, 170. Columbia University Quarterly, 242. Comptes Rendus, 344. Cornhill Magazine, 279.

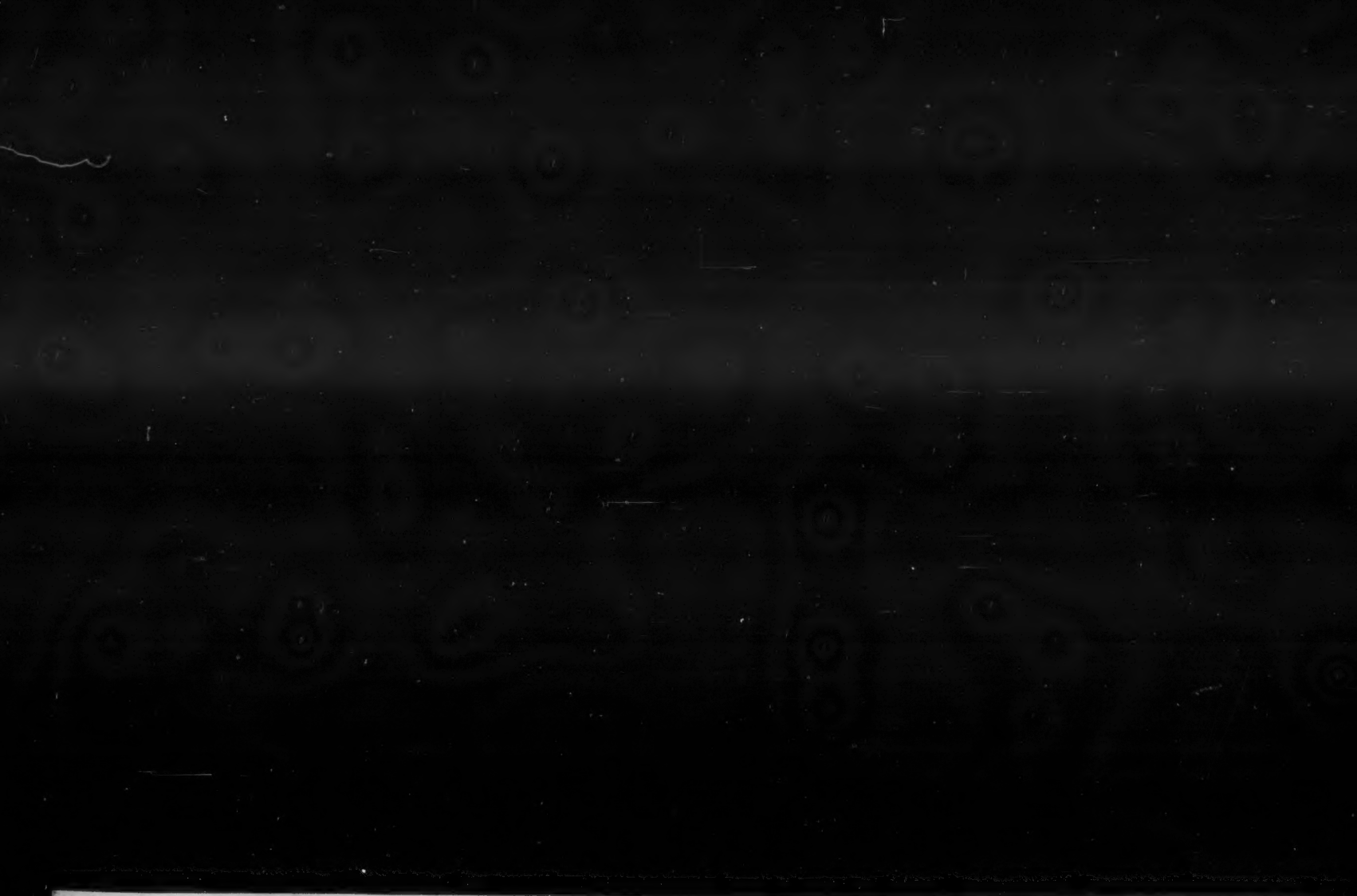
Dartmouth College, 168.
Educational Review, 203.
French Academy of Science, 312.
German Mathematical Society, 66. Giornali di Matematiche di Battaglini, 66.
Harvard Graduate Magazine, 310. Harvard University, 103. Hilbert Journal, 102.
International Congress of Mathematicians, 312. International Committee on the Teaching of Mathematics, 67, 246, 344. Irish Journal of Education, 66, 168, 205. Isis, 135, 170, 244.
Jahrbuch über die Fortschritte der Mathematik, 312.
L'Enseignement Mathematique, 169, 347. L'Intermediaire, 281.
Mathematical Gazette, 34, 35, 171, 279, 311. Mathematical Magazine, 66. Mathematische Annalen, 242. Mathematics Teacher, 243, 245, 311. Mathematics Teachers in New England, 206. Memorie della R. Accad. di Scienze, 103. Merchistonian, 311. Michigan Schoolmasters' Club, 206. Monist, 32.
Napier Tercentenary Celebration, 136, 208. National Educational Association, 169.
Open Court, 102, 244.
Paris Academy of Sciences, 104. Pennsylvania State College, 313. Philosophical Society of France, 68. Popular Science Monthly, 310. Proceedings of the American Philosophical Society, 311.
Quarterly Journal of Pure and Applied Mathematics, 204.
Rendiconti di Palermo, 279, 312. Revista de la Sociedad Matemática Española, 243, 280. Revue Semestralle, 312.
School Science and Mathematics, 32, 103, 135, 242. School Review, 32, 135. Science, 66, 170, 243, 281, 282. Sigma Xi, 169. Society for the Promotion of Engineering Education, 207, 345. Southwestern Section of the American Mathematical Society, 33, 343.
Tôkyô Sûgaku-Buturigakkwai Kizi, 67.
Universities of Chicago, 205, 280, 282, 343; Illinois, 168; Michigan, 168; Western Reserve, 135; Columbia University, 170.
Zeitschrift, 67, 171, 243.

SPECIAL ANNOUNCEMENT.

As important evidence that the MONTHLY will maintain in 1915 the same high standard which has been set in the two years since its reorganization, we call attention to the following announcement:

Under the title " History of Zeno's Arguments on Motion," Professor Cajori will begin in the January, 1915, issue a series of articles which promise to be of unusual interest. Our present well-organized theory of limits, like every modern system of thought in all fields, has its roots far back in human history. Just as the ideas of negative and imaginary numbers struggled for recognition and formulation through hundreds of years, so the notion of a variable approaching a limit and all of the far-reaching consequences connected with the consideration of such a concept have gone through a process of slow development. Professor Cajori traces this development through a two-thousand-year struggle for light in which practically all of the great mathematicians and philosophers of the time were engaged. These articles will begin with the Volume XXII and will extend well through the year 1915. The editors are greatly pleased that we are able to offer such an attraction to the readers of the MONTHLY for the coming year. No teacher who has to deal with the theory of limits can afford to miss this latest product of Professor Cajori's historical researches.

The editors would suggest that you can do a real service to those in your acquaintance by calling their attention to this unique opportunity and sending their names to the Managing Editor for sample copies.



INDEX TO VOLUME XXII

FORMAL PAPERS

ALTSHILLER, N. On the Circles of Apollonius, 261, 304.

AMES, L. D. The Probable Rank in a Large Class of a Student of given Rank in a Small Class, 334.

ARCHIBALD, R. C. Centers of Similitude of Circles and Certain Theorems Attributed to Monge, 6.

BILL, E. G. See REVIEWS, under Snyder.

BUSSEY, W. H. See REVIEWS under Crawley, Eliot, Dooley, Rivenburg, and Tuttle.

CAJORI, F. The History of Zeno's Arguments on Motion, 1, 39, 77, 109, 143, 179, 215, 253, 292.

CARMICHAEL, R. D. See REVIEWS under Russell.

CRAIG, C. F. See REVIEWS under Wentworth.

DICKSON, L. E. A Tribute to Mildred Lenora Sanderson, 264.

FORAKER, F. A. See REVIEWS under Thompson.

FORSYTH, C. H. A General Formula for the Valuation of Bonds, 149.

GATES, F. W. See REVIEWS under Williams and Robbins.

GLOVER, J. W. A General Formula for the Valuation of Securities, 82.

GRIFFIN, F. L. An Experiment in Correlating Freshman Mathematics, 325.

HEBBERT, C. M. A Cardiograph, 12.

JOHNSTON, W. A. See REVIEWS under Slocum.

KARPINSKI, L. C. See REVIEWS under Lobatschevski and Hill.

KEMPNER, A. J. See REVIEWS under Carmichael.

KEYSER, C. J. See REVIEWS under Manning.

KNOTT, C. G. The Theorem of Moments, 194.

LEAVENS, Dickson H. Linkages, 330.

LOVITT, W. V. A Geometrical Interpretation of Green's Formula, 152.

MILLER, G. A. Groups of Subtraction and Division with Respect to a Modulus, 47.

——— See REVIEWS under Dickson.

——— History of Mathematics, 299.

MOLINA, E. C. An Interpolation Formula for Poisson's Exponential Binomial Limit, 223.

MORITZ, R. On Napier's Fundamental Theorem Relating to Right Spherical Triangles, 220.

MOULTON, E. J. See REVIEWS under Dowling.

NICHOLSON, J. W. A Simple Solution of the Diophantine Equation $U^3=V^3+X^3+Y^3$, 224.

PASCAL, E. On a Certain Class of Determinants, 154.

PORTER, M. B. The Theory of Poles and Polars, 259.

RAYWORTH, J. C. See REVIEWS under Bôcher.

SHAW, J. B. See REVIEWS under Couturat.

SLAUGHT, H. E. The Promotion of Collegiate Mathematics, 251.

——— The Teaching of Mathematics, 289.

SNYDER, V. See REVIEWS under Slichter.

THOMPSON, H. D. The Identical Relations between the Direction Cosines of One Oblique Coördinate System Referred to Another Oblique System, 115.

WILSON, E. B. Linear Momentum, Kinetic Energy and Angular Momentum, 187.

——— See REVIEWS under Slichter.

PROBLEMS, QUESTIONS, AND DISCUSSIONS

AGARD, H. L. Problem solved in Calculus, 343.

ALTSHILLER, N. Problems proposed or solved in Algebra and Geometry, 20, 23, 24, 97, 101, 130, 135, 136, 137, 166, 206, 205, 230, 232, 274, 310, 314.

ATCHISON, C. S. Reply to Question relating to Conference Periods, 71.

ANNING, N. Discussion relating to a Geometric Representation of Integral Solutions of Certain Quadratic Equations, 321.

BAKER, R. P. Problems proposed in Geometry and Number Theory, 59, 60, 100, 207.

BALCH, J. V. Problem solved in Geometry, 168.

BARNETT, I. A. Problems proposed or solved in Calculus, 168, 169, 208, 209.

BARTON, V. Problem solved in Geometry, 165.

BEETLE, R. D. Problem solved in Calculus, 317.

BELL, E. T. Problems proposed in Geometry and Number Theory, 201, 202, 211.

BEMAN, W. W. Reply to Question in Calculus, 173; Problem solved in Calculus, 315.

BIGELOW, H. T. Problems solved in Algebra, Calculus, and Number Theory, 133, 314, 30.

BRENKE, W. C. Problem solved in Algebra, 19.

BROWN, B. J. Problems proposed or solved in Algebra, Geometry, Calculus and Mechanics, 24, 131, 161, 202, 230, 338.

BULLARD, J. A. Problem solved in Algebra, 99.

BURTON, W. W. Problems proposed or solved in Calculus, 267, 309, 340.

CAIRNS, W. D. Problems proposed or solved in Algebra, Calculus and Number Theory, 60, 67, 68, 26, 133, 228, 240, 267, 317, 338.

CAMPBELL, J. A. Problem proposed in Geometry, 338.